Math 461 F **Spring 2011**
Homework 6 Solutions **Drew Armstrong**

Book Problems.

Problem 6.4.8. Let r, s, t be the zeros of the real polynomial $x^3 + 23x + 1$. Find the real cubic (monic) polynomial whose zeros are r^2, s^2, t^2.

By assumption we have

$$\begin{aligned} x^3 + 23x + 1 &= (x-r)(x-s)(x-t) \\ &= x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst. \end{aligned}$$

Then equating coefficients on the left and right gives

$$\begin{aligned} -0 &= e_1 = r+s+t, \\ 23 &= e_2 = rs+rt+st, \\ -1 &= e_3 = rst. \end{aligned}$$

Now we are looking for the coefficients of the polynomial

$$\begin{aligned} f(x) &= (x-r^2)(x-s^2)(x-t^2) \\ &= x^3 - (r^2+s^2+t^2)x^2 + (r^2s^2+r^2t^2+s^2t^2)x - r^2s^2t^2. \end{aligned}$$

First note that $r^2s^2t^2 = e_3^2 = (-1)^2 = 1$.

Next (**Problem 6.4.1**) we wish to express $r^2+s^2+t^2$ in terms of e_1, e_2, e_3. Using Gauss' algorithm, or just trial and error, we have

$$(r^2+s^2+t^2) = (r+s+t)^2 - 2(r+s+t) = e_1^2 - 2e_2 = (-0)^2 - 2 \cdot 23 = -46.$$

Next (**Problem 6.4.4**) we wish to express $X = r^2s^2 + r^2t^2 + s^2t^2$ in terms of e_1, e_2, e_3. We will use Gauss' algorithm explicitly this time. First note that the leading term is r^2s^2. To eliminate this we will subtract e_2^2 which has the same leading term. This gives

$$X - e_2^2 = -2r^2st - 2rs^2t - 2rst^2 = -2(r+s+t)(rst) = -2e_1e_3.$$

Hence $X = e_2^2 - 2e_1e_3 = 23^2 - 2(-0)(-1) = 529$.

Finally, we conclude that the real cubic (monic) polynomial with roots r^2, s^2, t^2 is

$$f(x) = x^3 + 46x^2 + 529x - 1.$$

Additional Problems.

A.1. Euler showed that every real polynomial of the form $x^4 + \alpha x^2 + \beta x + \gamma$ factors into two real quadratics. Use his result to **prove** that every real polynomial of the form $ax^4 + bx^3 + cx^2 + dx + e$ factors into two real quadratics. (Hint: Use a change of variables to turn your polynomial into Euler's polynomial.)

Let $f(x) = ax^4 + bx^3 + cx^2 + dx = e$, where $a, b, c, d, e \in \mathbb{R}$ with $a \neq 0$. We wish to show that $f(x)$ factors as the product of two real quadratics. First note that $f(x - b/4a)$ is a real quartic polynomial with no x^3 term and leading coefficient a. We can make the leading coefficient 1 by dividing the whole thing by a. That is, we have

$$\frac{1}{a} f\left(x - \frac{b}{4a}\right) = x^4 + \alpha x^2 + \beta x + \gamma$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. Now Euler showed that this polynomial can be factored into two real quadratics, say $g(x)$ and $h(x) \in \mathbb{R}[x]$:

$$\frac{1}{a} f\left(x - \frac{b}{4a}\right) = g(x)h(x).$$

Finally, we undo the change of variables to obtain an explicit factorization of $f(x)$ into two real quadratics:

$$f(x) = a\ g\left(x + \frac{b}{4a}\right) h\left(x + \frac{b}{4a}\right).$$

A.2. Suppose that a given real quartic equation has roots a, b, c, d in some field $\mathbb{E} \supseteq \mathbb{R}$. (Today we know that these roots must be complex, but in times past their nature was mysterious.) Now let $f(a, b, c, d)$ be some function of a, b, c, d that is invariant under any permutation of the roots. Explain why $f(a, b, c, d)$ is a **real** number. (Hint: The Fundamental Theorem of Symmetric Functions.)

Suppose $f(a, b, c, d)$ is a symmetric function in a, b, c, d. Specifically, $f(a, b, c, d)$ is a polynomial in a, b, c, d with real (probably rational) coefficients which is invariant under any permutation of the inputs. By the Fundamental Theorem of Symmetric Functions, we can write

$$f = g(e_1, e_2, e_3, e_4),$$

for some unique polynomial g. By Gauss' algorithm, we know that the coefficients of g come from the coefficients of f, so they will also be real. Finally, we know that $-e_1, e_2, -e_3, e_4$ are the coefficients of the polynomial with roots a, b, c, d, and these coefficients are assumed to be be **real**. Thus $f = g(e_1, e_2, e_3, e_4)$ is real.

A.3. Let a, b, c, d be the roots of some real quartic equation with no x^3 term (i.e. we have $a+b+c+d = 0$.) Let $p = a+b$, $q = a+c$, and $r = a+d$, so that $-p = c+d$, $-q = b+d$, and $-r = b+c$. **Prove** that pqr is a real number, and hence $-p^2q^2r^2$ is a **negative** real number. (Hint: Show that pqr is invariant under permuting $a \leftrightarrow b$, or $a \leftrightarrow c$, or $a \leftrightarrow d$. Hence it's invariant under **any** permutation of a, b, c, d.)

By **Problem A.2.**, it is enough to show that pqr is invariant under any permutation of the inputs a, b, c, d. First, let's examine what happens if we switch the inputs $a \leftrightarrow b$. In this case $p \to p$, $q \to -r$ and $r \to -q$. Hence pqr turns into $p(-r)(-q) = pqr$. **It doesn't change.** Next, if we switch $a \leftrightarrow c$ we get $p \to -r$, $q \to q$ and $r \to -p$, so pqr turns into $(-r)q(-p) = pqr$. **It doesn't change.** Finally, what happens when we switch $a \leftrightarrow d$? We have $p \to -q$, $q \to -p$ and $r \to r$, hence pqr turns into $(-q)(-p)r = pqr$. **It doesn't change.** You could try a fex more permutations (for example, try $a \to b \to c \to d \to a$) and you will observe that pqr always remains the same. There are 24 permutations of a, b, c, d and you can check that they all leave pqr invariant, but in fact the three switches $a \leftrightarrow b$, $a \leftrightarrow c$ and $a \leftrightarrow d$ are enough because they generate all the others (you can take my word for it.) So we're done.

Since pqr is a symmetric function of a, b, c, d, and since a, b, c, d are the roots of a **real** polynomial, we conclude by **A.2.** that pqr is real. Hence $-p^2q^2t^2$ is a negative real number. (It could possibly be zero, but I don't want to worry about that. Lagrange took care of that case with a change of variables.)

Note: Problems A.1, A.2 and A.3 fill in some of the details in Euler's proof of the FTA. There are still a couple more details to fill, but they're not too hard. Lagrange and Kronecker took care of it for us.