Math 461 F **Spring 2011**
Homework 5 **Drew Armstrong**

Reading.

Chapter 6

Problems.

A.1. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$. If n is **even**, with $a_n > 0$ and $a_0 < 0$, **prove** that $f(x)$ has at least two real roots. (Hint: Intermediate value theorem.)

A.2. Leibniz (1702) claimed that $x^4 + a^4$ (for $a \in \mathbb{R}$) cannot be factored over $\mathbb{R}$. (In modern language, he claimed that $x^4 + a^4 \in \mathbb{R}[x]$ is irreducible.) **Prove him wrong.** (Hint: What are the fourth roots of $-a^4$?)

A.3. Nicolaus Bernoulli (1742) claimed in a letter to Euler that

$$f(x) = x^4 - 4x^3 + 2x^2 + 4x + 4$$

does not factor over $\mathbb{R}$. Euler responded (1743) that $f(x)$ has roots $1 \pm \alpha/2$ and $1 \pm \overline{\alpha}/2$, where

$$\alpha = \sqrt{2\sqrt{7}+4} + i\sqrt{2\sqrt{7}-4}.$$

Use this information to **prove Bernoulli wrong**.

A.4. Given a polynomial $p(x) \in \mathbb{C}[x]$ with complex coefficients, we define its conjugate polynomial $\overline{p}(x)$ by

$$\overline{p}(z) := \overline{p(\overline{z})} \qquad \text{for all } z \in \mathbb{C}.$$

This has the effect of conjugating the coefficients. **Prove** that the polynomial $f(x) = p(x)\overline{p}(x)$ has **real** coefficients.

For the following problems you should use Proposition 6.10 in the text, which says: If $G(x)$ is a greatest common divisor (common divisor with **largest degree**) of $A(x)$ and $B(x)$ over some field $\mathbb{F}$, then **there exist** polynomials $M(x)$ and $N(x)$ over $\mathbb{F}$ such that

$$A(x)M(x) + B(x)N(x) = G(x).$$

A.5. Prove: If $H(x)$ is any other common divisor of $A(x)$ and $B(x)$ then $H(x)$ divides $G(x)$. If $H(x)$ also has largest degree, then $H(x) = cG(x)$ for some nonzero constant $c \in \mathbb{F}$. Hence we can say that "the" greatest common divisor of $A(x)$ and $B(x)$ is **unique** up to nonzero constant multiples.

A.6. Euclid's Lemma for Polynomials. Let $P(x)$ be an irreducible polynomial over $\mathbb{F}$ (it cannot be factored into two polynomials of positive degree over $\mathbb{F}$) and suppose that $P(x)$ divides a product $F(x)G(x)$. In this case, **prove** that $P(x)$ must divide either $F(x)$ or $G(x)$ (or both).