Math 461 F — **Spring 2011**
Homework 3 Solutions — **Drew Armstrong**

Problems.

A.1. How many different (complex) numbers does the expression $\sqrt{1+\sqrt{3}}$ represent? Find a polynomial over $\mathbb{Z}$ which has these numbers as its roots.

The symbol $\sqrt{3}$ represents two numbers and thus $1+\sqrt{3}$ represents two numbers (both nonzero). Each of these numbers in turn has two square roots, so the symbol $\sqrt{1+\sqrt{3}}$ represents four distinct numbers. Let's look for an equation that these numbers must satisfy. Let x represent any value of the expression $\sqrt{1+\sqrt{3}}$. Then

$$\begin{aligned} x &= \sqrt{1+\sqrt{3}} \\ x^2 &= 1+\sqrt{3} \\ x^2-1 &= \sqrt{3} \\ (x^2-1)^2 &= 3 \\ x^4-2x^2-2 &= 0. \end{aligned}$$

Note that this last equation has at most four solutions. Hence it has **exactly** four solutions: the numbers $\sqrt{1+\sqrt{3}}$.

A.2. Use the trigonometric identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ together with Cardano's formula to find an expression for $\cos(\pi/9)$. (Note: This expression **must** involve complex numbers because $\cos(\pi/9)$ is not constructible.)

Put $\theta = \pi/9$ into the equation $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ to get the equation $1/2 = \cos(\pi/3) = 4\cos^3(\pi/9) - 3\cos(\pi/9)$. Letting $x = \cos(\pi/9)$ we get

$$4x^3 - 3x - \frac{1}{2} = 0, \quad \text{or} \quad x^3 - \frac{3}{4}x - \frac{1}{8} = 0.$$

Now we apply Cardano's formula (page 5 in the text) to get

$$\begin{aligned} \cos(\pi/9) = x &= \sqrt[3]{\frac{1}{16} + \sqrt{\frac{1}{16^2} - \frac{4}{16^2}}} - \sqrt[3]{-\frac{1}{16} + \sqrt{\frac{1}{16^2} - \frac{4}{16^2}}} \\ &= \sqrt[3]{\frac{1}{16} + \frac{1}{16}\sqrt{-3}} - \sqrt[3]{-\frac{1}{16} + \frac{1}{16}\sqrt{-3}} \\ &= \sqrt[3]{\frac{1+i\sqrt{3}}{16}} - \sqrt[3]{\frac{-1+i\sqrt{3}}{16}} \\ &= \frac{1}{\sqrt[3]{16}}\left(\sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}}\right). \end{aligned}$$

Actually, this formula represents three different numbers, one of which is $\cos(\pi/9)$.

A.3. Suppose that $p = 2^a + 1$ is a prime number. Show that a must be a power of 2. (Hint: If a has an **odd** factor b, show that the polynomial $x^b + 1$ factors nicely.)

Let $2^a + 1$ be a prime number. In this case we wish to show that a must be a power of 2. To prove this, let us assume the opposite. That is, **assume** that a is **not** a power of 2. This means that a must be divisible by an odd number, say $a = nb$ with b odd and $b \neq 1$ (why?). Hence we have $2^a + 1 = 2^{nb} + 1 = (2^n)^b + 1$. But now recall the formula for a difference of b-th powers:

$$1 - x^b = (1 - x)(1 + x^2 + x^3 + \cdots + x^{b-1}).$$

Since b is odd, we may replace x by $-x$ to get

$$1 + x^b = (1 + x)(1 - x + x^2 - x^3 + \cdots + x^{b-1}).$$

Finally, we conclude that

$$2^a + 1 = (2^n)^b + 1 = (1 + 2^n)(1 - 2^n + 2^{2n} - 2^{3n} + \cdots + 2^{(b-1)n}).$$

That is, we have expressed $2^a + 1$ as a product of two integers, neither of them equal to ± 1 (why?). This contradicts the fact that $2^a + 1$ is prime, and hence our assumption that a is **not** a power of 2 must be false. ///

A.4. Prove that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}.$$

(You can assume that the expression on the right converges.) We can describe this process **recursively** by setting $s_0 = 1$ and $s_n = 1 + 1/(1 + s_{n-1})$ for $n \geq 1$. What is s_4? How close is this to $\sqrt{2}$?

Since the expression on the right converges, it equals **some** number x, which must satisfy

$$x = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} = 1 + \frac{1}{1 + \left(1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}\right)} = 1 + \frac{1}{1 + x}.$$

Hence $x = 1 + 1/(1 + x)$, or $x^2 = 2$. Thus x is either the positive or negative square root of 2. We'll assume — without proof — that's it's the **positive** square root. (How could it be otherwise?)

Let's compute some partial... (what are they?) continued fractions, to see how **fast** they converge to $\sqrt{2}$. We have $s_0 = 1$, $s_1 = 1 + 1/(1 + 1) = 3/2$, $s_2 = 1 + 1/(1 + 3/2) = 7/5$, $s_3 = 1 + 1/(1 + 7/5) = 17/12$, and finally

$$s_4 = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = 1 + \frac{1}{1 + \frac{17}{12}} = \frac{41}{29} \approx 1.4138,$$

which agrees with $\sqrt{2} \approx 1.4142$ to three places.

Let D be the set of numbers that can be formed from $1, +, -, \times, \div, \sqrt{}$ in a finite number of steps. Now suppose that (x, y) is an intersection point for two circles

$$(x - a)^2 + (y - b)^2 = R^2, \tag{1}$$

$$(x - c)^2 + (y - d)^2 = r^2, \tag{2}$$

where $a, b, c, d, r, R \in D$. We want to show that $x, y \in D$.

A.5. What change of variables $(x', y') \to (x, y)$ **translates** the plane by $(-a, -b)$ and then **rotates** the plane by $-\tan^{-1}((d - b)/(c - a))$? Notice that we have $x', y' \in D \Leftrightarrow x, y \in D$. (I'll solve this one; you don't need to hand it in.)

We want a transformation that will send the circle centers (a, b) and (c, d) to $(0, 0)$ and $(\alpha, 0)$, respectively, for some $\alpha > 0$. First let T be the translation $T(x, y) = (x - a, y - b)$, so that $T(a, b) = (0, 0)$ and $T(c, d) = (c - a, d - b)$. Now we want to rotate the point $(c-a, d-b)$ onto the x-axis. This is accomplished by letting $\theta = -\tan^{-1}((d-b)/(c-a))$. To save space, let $\Delta := \sqrt{(c-a)^2 + (d-b)^2}$. Then the required rotation matrix is

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} c-a & d-b \\ -(d-b) & c-a \end{pmatrix}.$$

Applying R to $(0, 0)$ just gives $(0, 0)$ back, as desired. If we apply R to $(c - a, d - b)$ we get

$$\begin{aligned} R\begin{pmatrix} c-a \\ b-d \end{pmatrix} &= \frac{1}{\Delta}\begin{pmatrix} c-a & d-b \\ -(d-b) & c-a \end{pmatrix}\begin{pmatrix} c-a \\ d-b \end{pmatrix} \\ &= \frac{1}{\Delta}\begin{pmatrix} (c-a)^2 + (d-b)^2 \\ -(c-a)(d-b) + (c-a)(d-b) \end{pmatrix} \\ &= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}. \end{aligned}$$

Thus we have $R(T(a, b)) = (0, 0)$ and $R(T(c, d)) = (\alpha, 0)$, where $\alpha = \Delta$ (which equals the distance between the two circle centers), and RT is the desired transformation.

Now, if (x, y) is a point on the original two circles, then $R(T(x, y))$ will be a point on the transformed circles. We will call this new point (x', y'):

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} := RT\begin{pmatrix} x \\ y \end{pmatrix} &= R\begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ &= \frac{1}{\Delta}\begin{pmatrix} c-a & d-b \\ -(d-b) & c-a \end{pmatrix}\begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ &= \frac{1}{\Delta}\begin{pmatrix} (x-a)(c-a) + (y-b)(d-b) \\ -(x-a)(d-b) + (y-b)(c-a) \end{pmatrix}. \end{aligned}$$

Note that we can also express (x, y) in terms of (x', y') by applying the inverse transformation: $(x, y) = T^{-1}(R^{-1}(x', y'))$, where T^{-1} is translation by $(+a, +b)$ and R^{-1} is rotation by $+\theta$. For fun, I'll write this out:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}R^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} &= T^{-1}\frac{1}{\Delta}\begin{pmatrix} c-a & -(d-b) \\ d-b & c-a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= \frac{1}{\Delta}\begin{pmatrix} x'(c-a) - y'(d-b) \\ x'(d-b) + y'(c-a) \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned}$$

The actual formula here is not so important. The point I want to emphasize is that **the transformation $(x, y) \to (x', y')$ and its inverse $(x', y') \to (x, y)$ both preserve the property of being in the set D because they can be computed from the given quantities using $+, -, \times, \div, \sqrt{\ }$.**

A.6. Applying this transformation sends the center of circle (1) to the origin and then rotates the center of circle (2) to the x-axis, yielding a new system

$$x'^2 + y'^2 = R^2, \tag{3}$$

$$(x' - \alpha)^2 + y'^2 = r^2, \tag{4}$$

where $\alpha \in D$. Show that $x', y' \in D$, and hence $x, y \in D$.

For extra fun (not really — it's hell), you can plug the formula $(x, y) = T^{-1}(R^{-1}(x', y'))$ into equations (1) and (2) to observe that you get (3) and (4). (I did this on a computer, just to check. Yes, it works.) However, this is not necessary because we know for geometric reasons that if (x, y) is on the circles (1) and (2) then (x', y') is on the circles (3) and (4). So we just need to solve the new system.

First write $y'^2 = R^2 - x'^2$ and plug this into (4) to get

$$\begin{aligned} (x' - \alpha)^2 + R^2 - x'^2 &= r^2 \\ x'^2 - 2\alpha x' + \alpha^2 + R^2 - x'^2 &= r^2 \\ -2\alpha x' &= r^2 - R^2 - \alpha^2 \\ x' &= \frac{\alpha^2 + R^2 - r^2}{2\alpha}. \end{aligned}$$

Then plug back into (3) to get

$$y' = \pm\sqrt{R^2 - \left(\frac{\alpha^2 + R^2 - r^2}{2\alpha}\right)^2}.$$

This gives us two points of intersection. **Note that $x', y' \in D$, and hence $x, y \in D$, which is what we wanted to show. Done.**

Wait, do you want to see what the intersection points look like in (x, y) coordinates? Okay, here they are. (Recall $\alpha = \Delta$.) The first point of intersection is

$$\begin{aligned} x &= \frac{(c-a)\left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right) + (d-b)\sqrt{R^2 - \left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right)^2}}{\Delta} + a, \\ y &= \frac{(d-b)\left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right) - (c-a)\sqrt{R^2 - \left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right)^2}}{\Delta} + b. \end{aligned}$$

and the second is

$$\begin{aligned} x &= \frac{(c-a)\left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right) - (d-b)\sqrt{R^2 - \left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right)^2}}{\Delta} + a, \\ y &= \frac{(d-b)\left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right) + (c-a)\sqrt{R^2 - \left(\frac{\Delta^2+R^2-r^2}{2\Delta}\right)^2}}{\Delta} + b. \end{aligned}$$