Math 311 **Spring 2023**
Class Summary **Drew Armstrong**

Contents

January

Wed, Jan 18

- The first differential equation. Let $h(t)$ be height of a stone near surface of Earth. Galileo: $h''(t) = -32$ feet per second squared. Integrate twice to get $h(t) = -16t^2 + v_0 t + h_0$. Use this to predict when the stone will land.

- Solutions to differential equations are built from a small number of basic functions. Most important: $dy/dx = y$ has solution $y(x) = e^x$. $d^2y/dx^2 = y$ has solution $y(x) =$"sin(x)". Actually, not quite. Equation $d^2y/dx^2 = -y$ has solution $\sin(x)$. Equation $d^2y/dx^2 = -k^2y$ has solution $y(x) = \sin(kx)$. Put $-k^2 = 1$ (i.e., k=i) to get the solution of $d^2y/dx^2 = y$. This shows that imaginary numbers sometimes show up in the solutions to real problem. More later.

Fri, Jan 20

- Recall $x(t) = e^{kt}$ satisfies $x'(t) = kx(t)$ and $x(t) = \sin(kt)$ satisfies $x''(t) = -k^2x(t)$.

- General first order differential equation $dy/dx = f(x, y)$ always has a one parameter family of solutions. E.g. general solution of $dy/dx = x$ is $y(x) = x^2/2 + C$. General solution of $dy/dx = y$ is $y(x) = Ce^{kx}$.

- One initial condition $y(a) = b$ determines the value of the parameter C.

- Geometry: Think of $dy/dx = f(x, y)$ as a slope field. Solution curves fit the slope.

- Remark: The equation $d^2y/dx^2 = -k^2y$ is second order. It has a two parameter family of solutions: $y(x) = C_1 \sin(kx) + C_2 \cos(kx)$. More later.

Mon, Jan 23

- Recall: First order equations $dy/dx = f(x, y)$ and slope fields.
- Online slope field generator:

  `https://homepages.bluffton.edu/~nesterd/apps/slopefields.html`

- Easiest kind $dy/dx = f(x)$ solved by direct integration: $y(x) = \int f(x)\,dx + C$.
- Exponential growth $dy/dx = y$. Solution $y(x) = e^x$.
- Explosive growth $dy/dx = y^2$. Solution $y(x) = 1/(1-x)$ has a vertical asymptote.
- Separable equations have the form $dy/dx = g(x)h(y)$. We can solve by separation of variables: $dy/h(y) = g(x)dx$ and then

  $$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx + C.$$

  At the end, try to solve for y. It's not always possible.
- Example: $dy/dx = xy$. Solution $y(x) = Ae^{x^2/2}$.

Wed, Jan 25

- We have seen exponential growth $dx/dt = x$ and explosive growth $dx/dt = x^2$.
- More realistic: Logistic growth $dx/dt = x(1-x)$. Starts exponential but then settles down to carrying capacity. Use partial fractions:

  $$\begin{aligned}
  \int \frac{1}{x(1-x)}\,dx &= \int dt + C \\
  \int \left[\frac{1}{x} - \frac{1}{1-x}\right] dx &= t + C \\
  \ln(x) - \ln(1-x) &= t + C \\
  \ln\left(\frac{x}{1-x}\right) &= t + C \\
  \frac{x}{1-x} &= e^{t+C} = e^C e^t = De^t \qquad \text{(clean up: } D = e^C\text{)} \\
  x(t) &= \frac{De^t}{1+De^t}.
  \end{aligned}$$

  Sketch. Note: $x(t) \to D$ as $t \to \infty$. Given initial population size $x(0) = x_0$ we get

  $$x_0 = \frac{De^0}{1+De^0}$$

$$x_0 = \frac{D}{1+D}$$
$$D = \frac{x_0}{1-x_0},$$

so that

$$x(t) = \frac{\left(\frac{x_0}{1-x_0}\right)e^t}{1+\left(\frac{x_0}{1-x_0}\right)e^t}.$$

There are other ways to write this. For example, multiply top and bottom by $1-x_0$:

$$x(t) = \frac{x_0 e^t}{1-x_0+x_0 e^t}.$$

Check: $x(0) = x_0/(1-x_0+x_0) = x_0$.

Fri, Jan 27

- General first order equation $dy/dx = f(x,y)$. Simplest examples of $f(x,y)$ are x, x^2, y, y^2, $x+y$, x^2+y, x^2+y^2. These last ones are not separable.
- The examples x^2+y and x^2+y^2 can't be solved with elementary functions. Someone invented "Bessel functions" to solve them.
- We will solve the simplest non-separable equation $x'(t) = x+t$. This is part of the general theory of "linear" differential equations and we will see several methods.
- The first method is "integrating factors". Multiply both sides by some suitable function of t. Let's call it $\rho(t)$:

$$x'(t) = x+t$$
$$\rho(t)x'(t) = \rho(t)(x+t)$$
$$\rho(t)x'(t) = \rho(t)x+\rho(t)t$$
$$\rho(t)x'(t)-\rho(t)x(t) = \rho(t)t.$$

- This trick is to choose $\rho(t)$ so the left side looks like the product rule. We want

$$\rho(t)x'(t)-\rho(t)x(t) = \varphi(t)x'(t)+\varphi'(t)x(t) = (\varphi(t)x(t))'$$

for some function $\varphi(t)$. Thus we must have $-\rho(t) = \rho'(t)$ and hence $\rho(t) = e^{-t}$.

- So let $\rho(t) = e^{-t}$. Finish the computation:

$$x'(t) = x+t$$
$$e^{-t}x'(t) = e^{-t}(x+t)$$

$$\begin{aligned} e^{-t}x'(t) - e^{-t}x(t) &= te^{-t} \\ (e^{-t}x(t))' &= te^{-t} \\ e^{-t}x(t) &= \int te^{-t}\,dt + C \end{aligned}$$

Off to the side we compute $\int te^{-t}\,dt = (-t-1)e^{-t}$ using integration by parts. Then

$$\begin{aligned} e^{-t}x(t) &= (-t-1)e^{-t} + C \\ x(t) &= -t - 1 + Ce^{-t}. \end{aligned}$$

That looks rather nice. Maybe there is an easier way to do it.

Mon, Jan 30

- Most differential equations are impossible to solve exactly. Instead we linearize them and apply linear algebra to run a numerical simulation. Today we take the first step into "linear differential equations".

- A first order *linear* ODE has the form $dy/dx + P(x)y = Q(x)$. We have seen several of these, e.g., $dy/dx = y$ and $dy/dx = f(x)$ for any function $f(x)$. We have also seen some non-linear equations such as $dy/dx = y^2$ and $dy/dx = y(1-y) = y - y^2$. Linear means only y and dy/dx appear. No terms such as y^2, $(dy/dx)^2$ or $y \cdot dy/dx$.

- The general n-th order linear ODE is

  $$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x),$$

  where $y^{(n)}$ is the nth derivative of y with respect to x and $f(x), p_1(x), \ldots, p_n(x)$ are any functions of x.

- Method of integrating factors. Multiply both sides by a suitable function $\rho(x)$:

  $$\begin{aligned} \frac{dy}{dx} + P(x)y &= Q(x) \\ \rho(x)\left(\frac{dy}{dx} + P(x)y\right) &= \rho(x)Q(x). \end{aligned}$$

  The **clever choice** is $\rho(x) = e^{\int P(x)\,dx}$ where $\int P(x)\,dx$ is any antiderivative of $P(x)$. This makes the left hand side look like the product rule. To be specific, the choice $\rho(x) = e^{\int P(x)\,dx}$ is clever because it satisfies $\rho'(x) = \rho(x)P(x)$. Thus we get

  $$\begin{aligned} \rho(x)\left(\frac{dy}{dx} + P(x)y\right) &= \rho(x)\frac{dy}{dx} + \rho(x)P(x)y \\ &= \rho(x)\frac{dy}{dx} + \rho'(x)y \\ &= \frac{d}{dx}\left[\rho(x) \cdot y\right]. \end{aligned}$$

- Example: $x^2 \cdot dy/dx + x \cdot y = \sin(x)$. First divide by x^2 to put in standard form:

$$y'' + \frac{1}{x}y = \frac{\sin(x)}{x^2}.$$

Thus $P(x) = x$ and $Q(x) = \sin(x)$. The integrating factor is

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int dx/x} = e^{\ln(x)} = x.$$

Multiply both sides by x then integrate:

$$\begin{aligned} y'' + y/x &= \sin(x)/x^2 & \\ xy'' + y &= \sin(x)/x & \text{multiply both sides by } x \\ (xy)' &= \sin(x)/x & \text{product rule} \\ xy &= \int \frac{\sin(x)}{x}\,dx + C & \text{integrate} \\ y &= \frac{1}{x}\left[\int \frac{\sin(x)}{x}\,dx + C\right] & \end{aligned}$$

More later.

February

Wed, Feb 1

- Remark about the homework: The integral $\int e^{-x^2}\,dx$ can't be simplified. But it's really important in statistics so we give it a name. It is typically called the *error function*:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\,ds.$$

The constant multiple $2/\sqrt{\pi}$ is not important for us but it is part of the standard notation. Thus for any limits of integration $a < b$ we can write

$$\int_a^b e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\left[\text{erf}(b) - \text{erf}(b)\right].$$

You **do not need to use this notation**. Just be aware that computers use it.

- Recall integrating factors for first order ODEs. To solve $dy/dx + P(x)y = Q(x)$, multiply both sides by the integrating factor

$$\rho(x) = e^{\int P(x)\,dx} = \exp\left(\int P(x)\,dx\right).$$

Then the left hand side simplifies to $(d/dx)[\rho(x)y]$, via the product rule. (This is the trick!) So the general solution is

$$\begin{aligned}\frac{dy}{dx} + P(x)y &= Q(x) \\ \rho(x)\left(\frac{dy}{dx} + P(x)y\right) &= \rho(x)Q(x) \\ \frac{d}{dx}\left[\rho(x)y\right] &= \rho(x)Q(x) \\ \rho(x)y &= \int \rho(x)Q(x)\,dx + C \\ y &= \frac{1}{\rho(x)}\left[\int \rho(x)Q(x) + C\right].\end{aligned}$$

- Textbook example, page 51. Last time we showed that the linear ODE $x^2 \cdot dy/dx + x \cdot y = \sin(x)$ has integrating factor x and **general solution**

$$y = \frac{1}{x}\left[\int \frac{\sin x}{x}\,dx + C\right]$$

To find a **specific solution**, suppose we have initial condition $y(1) = y_1$. (It doesn't make sense to put $x = 0$ because the solution has $1/x$ in it, so we will always assume that $x > 0$.) Now in order to solve for C in terms of y_1 we need to turn the indefinite integral as a definite integral. We do this by choosing an **arbitrary lower limit** a:

$$y(x) = \frac{1}{x}\left[\int_a^x \frac{\sin s}{s}\,ds + C\right].$$

Now put $x = 1$ to get

$$y_1 = y(1) = \frac{1}{1}\left[\int_a^1 \frac{\sin s}{s}\,ds + C\right] = \int_a^1 \frac{\sin s}{s}\,ds + C,$$

and hence

$$C = y_1 - \int_a^1 \frac{\sin s}{s}\,ds.$$

Again, the lower limit a is completely arbitrary. We see from this formula for C that the most clever choice is $a = 1$, so that

$$C = y_1 - \int_1^1 \frac{\sin s}{s}\,ds = y_1 - 0 = y_1,$$

and the specific solution with $y(1) = y_1$ is

$$y(x) = \frac{1}{x}\left[\int_1^x \frac{\sin s}{s}\,ds + y_1\right].$$

The integral does not simplify. Your computer will probably say

$$y(x) = \frac{1}{x}\left[\mathrm{Si}(x) - \mathrm{Si}(1) + y_1\right],$$

where the *Sine integral* is defined as follows:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(s)}{s}\, ds.$$

- We did textbook Example 2 on page 49. The first order linear ODE

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x$$

has **general solution**

$$y(x) = \frac{1}{(x^2+1)^{3/2}}\left[\int 6x(x^2+1)^{1/2}\, dx + C\right] = 2(x^2+1)^{3/2} + C.$$

We didn't mess around with initial conditions.

Fri, Feb 3

- We discussed Homework 1. Please read the homework solutions!
- VERY IMPORTANT REMINDER:

$$e^t \to \infty \text{ as } t \to \infty,$$
$$e^{-t} \to 0 \text{ as } t \to \infty.$$

This is the difference between exponential growth and exponential decay. Many solutions to ODEs contain the term e^{-t}, which tells us something important about the behavior as $t \to \infty$.

- For example, the logistic equation $x(t) = x(1-x)$ has the following solution:

$$x(t) = \frac{Ce^t}{1+Ce^t} = \frac{1}{1+e^{-t}/C}.$$

So that

$$x(t) \to \frac{1}{1+0} = 1 \quad \text{as} \quad t \to \infty.$$

Mon, Feb 6

- Finished a problem from last Friday. Room temperature $A(t) = t$, coffee temperature $u(t)$. Newton says $u'(t) = A(t) - u(t)$. [Check: If coffee is colder than room then coffee heats up, $u'(t) > 0$. If coffee is hotter than room then coffee cools down, $u'(t) < 0$.] Method of integrating factors:

$$\begin{aligned}
u'(t) &= t - u(t) \\
u'(t) + u(t) &= t \\
u'(t) + P(t)u(t) &= Q(t) \\
e^t(u'(t) + u(t)) &= te^t && \text{integrating factor } \exp\left(\int P(t)\,dt\right) = e^t \\
e^t u'(t) + e^t u(t) &= te^t \\
(e^t u(t))' &= te^t && \text{yes, the method worked} \\
e^t u(t) &= \int te^t\,dt + C \\
e^t u(t) &= (t-1)e^t + C && \text{integration by parts omitted} \\
u(t) &= t - 1 + Ce^{-t}.
\end{aligned}$$

- Qualitative: For large t we have $e^{-t} \approx 0$, so $u(t) \approx t - 1$, which is one degree colder than the room. See homework 1 solutions for a picture.

- Qualitative: Suppose the coffee is in an insulated container. Then we have $u'(t) = k(t-u)$ for some insulation constant $k > 0$. Carry out the same method to get

$$u(t) = t - \frac{1}{k} + Ce^{-kt}.$$

  This time $u(t) \approx t - \frac{1}{k}$ for large t, so the coffee becomes $1/k$ degrees colder than the room. We drew some pictures.

- Introduction to Chapter 2. A first order equation $dy/dx = f(x, y)$ has a "one parameter family of solutions". For example $dy/dx = y$ has general solution $y = Ce^x$ or $y = e^{x+D}$ (with $D = e^C$). Second order equations have a "two parameter family of solutions".

- The most basic second order equation is $y'' = -y$. We have seen that $y = \sin(x)$ and $y = \cos(x)$ are solutions. I claim that the **general solution** is

$$y(x) = C_1 \cos(x) + C_2 \sin(x)$$

  for any constants C_1 and C_2. Check that this is, indeed, a solution. It is much more difficult to show that there are no **other** solutions; you can just trust me on this.

- Values of the parameters C_1 and C_2 are determined by **two** initial conditions; traditionally, $y(0)$ and $y'(0)$ (called initial position and initial velocity). In our case check that $C_1 = y(0)$ and $C_2 = y'(0)$:

$$y(x) = y(0)\cos(x) + y'(0)\sin(x).$$

Sketch the solutions $y(x)$ with $y(0) = 0, y'(0) = 1$ and $y(0) = 1, y'(0) = 0$. Observe that they have the correct initial position and initial slope.

Wed, Feb 8

- *Hooke's Law* says that

$$\text{Spring Force} \propto \text{Negative Displacement}$$
$$mx''(t) = -kx(t),$$

where $m > 0$ is the mass of the moving object and $k > 0$ is the stiffness of the spring. Here $x(t)$ is the displacement of the object from equilibrium. So the spring always wants to move $x(t)$ towards 0. Unlike a first order equation (such as heat), the spring equation causes overshoot and oscillation.

- For simplicity take $m = k = 1$. Then we saw that the solution is

$$x(t) = x(0)\cos t + x'(0)\sin t.$$

But trig functions are chameleons and there are many different ways to express the solution.

- It is more meaningful to write

$$x(t) = C\cos(t - \alpha),$$

where C is the *amplitude* and α is the *phase shift* of the oscillation.

- Problem: Given

$$A\cos t + B\sin t = C\cos(t - \alpha),$$

find the "change of parameters" equations between A, B and C, α.

- Option 1: **Memorize** the formulas $A = C\cos\alpha$ and $B = C\sin\alpha$. Then draw a right triangle and use it to get $C = \sqrt{A^2 + B^2}$ and $\alpha = \tan^{-1}(B/A)$.

- Option 2: *Memorize* the "angle difference formula":

$$\cos(t - \alpha) = \cos\alpha\cos t + \sin\alpha\sin t.$$

Then multiply both sides by C and compare to get $A = C\cos\alpha$ and $B = C\sin\alpha$:

$$\cos(t - \alpha) = \cos\alpha\cos t + \sin\alpha\sin t$$
$$C\cos(t - \alpha) = (C\cos\alpha)\cos t + (C\sin\alpha)\sin t.$$

- Option 3: *Memorize* Euler's identity:

$$e^{it} = \cos t + i\sin t.$$

Use this to get the angle difference formula, then the formulas for A and B.

Fri, Feb 10

- The spring equation: $mx''(t) = -kx(t)$. If we write $\omega = \sqrt{k/m}$ then we find the general solution
$$x(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega(t-\alpha)).$$
Here $\omega = \sqrt{k/m}$ is the circular frequency, the number of full rotations per second. Stiffer spring makes faster oscillations. Heavier object makes slower oscillations. Since a full rotation is 2π radians, the period (number of oscillations per second) is $T = 2\pi/\omega$.

- There is a homework problem about Euler's Identity: $e^{it} = \cos t + i\sin t$. You will use it to derive the angle sum/difference identities.

- Euler's identity is very important to the study of linear ODEs. For example, the general method we will learn for solving second order linear ODEs will take the equation $x''(t) = -x(t)$ and spit out the solution
$$x(t) = C_1 e^{it} + C_2 e^{-it}.$$
This is equivalent to the solution $x(t) = A\cos t + B\sin t$, but you need Euler's identity to see this.

- The general method to solve the second order linear ODE
$$x''(t) - 5x'(t) + 6x(t) = 0$$
is to substitute the test function $x(t) = e^{\lambda t}$. Doing this gives
$$\begin{aligned} x''(t) - 5x'(t) + 6x(t) &= 0 \\ \lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 - 5\lambda + 6) &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0. \end{aligned}$$
This quadratic equation has solutions $\lambda = 2$ and 3. Hence e^{2t} and e^{3t} are basic solutions, and combining them gives the general solution
$$x(t) = C_1 e^{2t} + C_2 e^{3t}.$$
Check that this works.

- Use the method on the spring equation:
$$mx''(t) + kx(t) = 0.$$
Guess the basic solution $x(t) = e^{\lambda t}$ to get $m\lambda^2 + k = 0$ and hence $\lambda = \pm\sqrt{-k/m}$. Hence the general solution is
$$x(t) = C_1 \exp(\sqrt{-k/m}\cdot t) + C_2\exp(-\sqrt{-k/m}\cdot t).$$
This is correct but it requires some interpretation because physically real solutions correspond to $k > 0$ and $m > 0$, so that $\sqrt{-k/m}$ is an imaginary number. This is where Euler's identity comes in.

Mon, Feb 13

- The spring equation without friction is $mx''(t) = -kx(t)$. The general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation.

- Now suppose there is some friction that is proportional to the velocity $x'(t)$. Then the spring equation becomes $x''(t) = -kx(t) - cx'(t)$, or

$$mx''(t) + cx'(t) + kx(t) = 0$$

for some friction (also called damping) constant $c > 0$. (Section 2.4 of the textbook.) This equation becomes much easier to solve if we use complex numbers.

- Review of (or introduction to) complex numbers: Let i be a square root of -1 so that $i^2 = -1$. A *complex number* has the form

$$a + ib \text{ for some real numbers } a, b.$$

The product of complex numbers is

$$(a+ib)(c+id) = \text{ FOIL details } = (ac - bd) + i(ad + bc).$$

- In order to divide, we multiply numerator and denominator by the *complex conjugate*:

$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \text{ FOIL details } = \left(\frac{a}{a^2+b^2}\right) + i\left(\frac{-b}{a^2+b^2}\right).$$

- These formulas look complicated. Everything simplifies when we express complex numbers in *polar form*. Draw $a + ib$ as the point (a, b) in the Cartesian plane. Let r be the distance from $(0, 0)$ to (a, b) and let θ be the angle from the "x-axis" to (a, b), so that $a = r\cos\theta$ and $b = r\sin\theta$. Then from Euler's identity we have

$$a + ib = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

- Multiplication is now much easier. Consider any two complex numbers $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$ expressed in polar form. Then

$$r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i\theta_1+i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}.$$

That is, to multiply complex numbers we *multiply their lengths and add their angles.*

- Application. Solve the equation $x^3 = 1$. We are looking for complex solutions $x = a+ib$. If you expand $(a+ib)^3$ then after a lot of trial and error you will find three solutions:

$$x = 1 \quad \text{or} \quad -\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{or} \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

These solutions are much easier to find if we express $x = re^{i\theta}$ in polar form. First of all, since $x^3 = 1$ we see that x^3 has length 1 so x must have length 1, i.e., $r = 1$. Then

$$\begin{aligned} x^3 &= 1 \\ (e^{i\theta})^3 &= 1 \\ e^{i(3\theta)} &= 1 \\ \cos(3\theta) + i\sin(3\theta) &= 1 \\ \cos(3\theta) + i\sin(3\theta) &= 1 + i0, \end{aligned}$$

so that $\cos(3\theta) = 1$ and $\sin(3\theta) = 0$. The first equation says that $3\theta = 2\pi k$ for some k and the second says that $3\theta = \pi n$ for some n. There are three solutions for θ:

$$\theta = 0 \quad \text{or} \quad \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi}{3}.$$

Hence there are three solutions for $x = e^{i\theta}$:

$$\begin{array}{rclllll} x &=& e^{i0} & \text{or} & e^{i2\pi/3} & \text{or} & e^{4\pi/3} \\ &=& 1 & \text{or} & \cos(2\pi/3) + i\sin(2\pi/3) & \text{or} & \cos(4\pi/3) + i\sin(4\pi/3) \\ &=& 1 & \text{or} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & \text{or} & -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{array}$$

So it was really a problem of trigonometry.

- The same method shows that the equation $x^n = 1$ has n distinct complex solutions:

  $$x = 1, e^{i2\pi/n}, e^{i4\pi/n}, \ldots, e^{i2\pi ik/n}, \ldots, e^{i2\pi(n-1)/n}.$$

  In the "complex plane" these are n points equidistributed around the unit circle. In cartesian coordinates, the kth solution is

  $$x = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right).$$

  Unlike with $n = 3$, for most values of n these numbers cannot be simplified.

- I apologize for this detour but it's necessary because complex numbers are **essential** to differential equations. In principle this is all in the pre-calculus curriculum but in my experience most students don't see it. It will make more sense as we use it in applications.

Wed, Feb 15

- Reminder: The solution to $x''(t) = -x(t)$ can be expressed in at least three ways:

  $$\begin{aligned} x(t) &= A\cos t + B\sin t \\ &= C\cos(t - \alpha) \\ &= De^{it} + Ee^{-it}. \end{aligned}$$

  The relationship between (A, B) and (C, α) is

  $$\left\{ \begin{array}{l} A = C\cos\alpha, \\ B = C\sin\alpha, \end{array} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{array}{l} C = \text{(homework)}, \\ \alpha = \text{(homework)}. \end{array} \right\}$$

- The relationship between (A, B) and (D, E) comes from Euler's identity $e^{it} = \cos t + i \sin t$. You will investigate this on Homework 2.

- Why do we bother with complex numbers in this course? Because they are necessary for understanding second order ODEs. Here is the general method: To solve

$$ax''(t) + bx'(t) + cx(t) = 0,$$

we guess the solution $x(t) = e^{\lambda = t}$. Then substitute and solve for λ:

$$\begin{aligned} ax''(t) + bx'(t) + cx(t) &= 0 \\ a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} &= 0 \\ e^{\lambda t}(a\lambda^2 + b\lambda + c) &= 0. \end{aligned}$$

Since $e^{\lambda t}$ is **never zero**, this implies that $a\lambda^2 + b\lambda + c = 0$. This quadratic equation for λ has two solutions, which we can obtain from the quadratic formula:

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Then the general solution to the ODE is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

- Example: To solve $x''(t) + x(t) = 0$ we substitute $x(t) = e^{\lambda t}$ to get

$$\begin{aligned} x''(t) + x(t) &= 0 \\ \lambda^2 e^{\lambda t} + e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 + 1) &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda &= \pm i. \end{aligned}$$

Hence the general solution is

$$x(t) = C_1 e^{it} + C_2 e^{-it} = C_3 \cos t + C_4 \sin t,$$

as we already knew.

- Using the same method, the undamped oscillator $mx''(t) + kx(t)$ has solution

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} = C_3 \cos(\omega t) + C_4 \sin(\omega t),$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation, which we also knew.

- Here is something new. The **damped oscillator** has equation $mx''(t) = -kx(t) - cx(t)$, where m is the inertia (which resists acceleration), k is the stiffness (which accelerates toward equilibrium) and c is the **friction** (which resists velocity).

- Example: To solve $x''(t) = -x(t) - x'(t)$ we substitute $x(t) = e^{\lambda t}$:

$$\begin{aligned} x''(t) + x'(t) + x(t) &= 0 \\ \lambda^2 e^{\lambda t} + \lambda e^{\lambda t} + e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 + \lambda + 1) &= 0 \\ \lambda^2 + \lambda + 1 &= 0 \\ \lambda &= \frac{-1 \pm \sqrt{-3}}{2} && \text{quadratic formula} \\ \lambda &= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \end{aligned}$$

Hence the general solution is

$$\begin{aligned} x(t) &= C_1 e^{(-1/2+i\sqrt{3}/2)t} + C_2 e^{(-1/2-i\sqrt{3}/2)t} \\ &= C_1 e^{-t/2} e^{i(\sqrt{3}/2)t} + C_2 e^{-t/2} e^{-i(\sqrt{3}/2)t} \\ &= e^{-t/2}\left[C_1 e^{i(\sqrt{3}/2)t} + C_2 e^{-i(\sqrt{3}/2)t}\right] \\ &= e^{-t/2}\left[C_3 \cos\left(\frac{\sqrt{3}}{2}\cdot t\right) + C_4 \sin\left(\frac{\sqrt{3}}{2}\cdot t\right)\right]. \end{aligned}$$

Note that the imaginary numbers played an important role in the solution. Interpretation: The spring oscillates with frequency $\sqrt{3}2$, while the amplitude quickly decays to zero because the whole thing is multiplied by $e^{-t/2}$. The decay part makes sense because there is friction. The exact value of the frequency is a bit of a surprise. Notice that $\sqrt{3}/2 \approx 0.87$, which is smaller than the frequency of the **undamped oscillator** $x''(t) + x(t) = 0$, which has frequency 1. Damping reduces the frequency.

- Too much damping: Plugging $e^{\lambda t}$ into the equation $x''(t) + 5x'(t) + 6x(t) = 0$ gives $\lambda = -2$ and $\lambda = -3$, so the general solution is

$$x(t) = C_1 e^{-3t} + C_2 e^{-3t}.$$

In this case the values of λ are **real** so there is **no oscillation**. And the values are **negative** so the solution **decays**. This happened because the friction was large.

- Qualitative Interpretation:

$$\begin{aligned} \text{imaginary part of } \lambda &\rightsquigarrow \text{oscillation}, \\ \text{real part of } \lambda &\rightsquigarrow \text{growth or decay}. \end{aligned}$$

- Quantitative: $mx''(t) + cx'(t) + kx(t) = 0$ has solution $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ where

$$\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

If $\boxed{c^2 - 4mk < 0}$ then the square root is imaginary, so there is oscillation. Otherwise there is no oscillation. If m, c, k are **positive** then the real parts of λ_1, λ_2 are always < 0 so the solution will **decay**. To make the solution grow you need one of m, c, k to be negative, which is non-physical.

Fri, Feb 17

- We discussed the Homework 2 solutions. Please read them: `https://www.math.miami.edu/~armstrong/311sp23/311sp23hw2sol.pdf`

Mon, Feb 20

- We have seen that second order linear ODEs correspond to oscillations with damping. This week instead of physics we will just blast through some textbook-style problems to develop calculation skills.

- Solve the equation $x''(t) + 2x'(t) + 2x(t) = 0$ with initial conditions $x(0) = 1$ and $x'(0) = \sqrt{3} - 1$. (This strange condition was chosen to make the answer work out nicely.) The general method for this type of equation is to substitute $x(t) = e^{\lambda t}$ and solve for λ. Note that $x(t) = e^{\lambda t}$ implies $x'(t) = \lambda e^{\lambda t}$ and $x''(t) = \lambda^2 e^{\lambda t}$. Thus we want to solve

$$\begin{aligned} x''(t) + 2x'(t) + 2x(t) &= 0 \\ \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 2e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 + 2\lambda + 2) &= 0 \\ \lambda^2 + 2\lambda + 2 &= 0. && \text{because } e^{\lambda t} \text{ is never zero} \end{aligned}$$

This is called the *characteristic equation* of $x''(t) + 2x'(t) + 2x(t) = 0$. According to the quadratic formula, the roots are

$$\lambda_1, \lambda_2 = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

Hence the general solution of $x''(t) + 2x'(t) + 2x(t) = 0$ is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t},$$

for some constants c_1 and c_2.

- To express this in terms of real numbers, we note that

$$e^{(-1+i)t} = e^{-t+it} = e^{-t}e^{it} \quad \text{and } e^{(-1-i)t} = e^{-t-it} = e^{-t}e^{-it},$$

  so that

$$\begin{aligned} x(t) &= c_1 e^{-t}e^{it} + c_2 e^{-t}e^{-it} \\ &= e^{-t}\left(c_1 e^{it} + c_2 e^{-it}\right). \end{aligned}$$

  Then substituting $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos t - i \sin t$ gives

$$x(t) = e^{-t}\left(c_3 \cos t + c_4 \sin t\right)$$

  for some new constants c_3 and c_4. (We could express c_3, c_4 in terms of c_1, c_2 but there is no reason to do this.)

- Finally, we will use the initial conditions $x(0) = 1$ and $x'(0) = \sqrt{3} - 1$ to solve for c_3 and c_4. First we have

$$1 = x(0) = e^0(c_3 \cos(0) + c_3 \sin(0)) = c_3.$$

  To input $x'(0)$ we first need to compute $x'(t)$ using the product rule:

$$\begin{aligned} x'(t) &= \frac{d}{dt}e^{-t}\left(c_3 \cos t + c_4 \sin t\right) \\ &= e^{-t}(-c_3 \sin t + c_4 \cos t) - e^{-t}(c_3 \cos t + c_4 \sin t). \end{aligned}$$

  Then we substitute $x'(0) = \sqrt{3} - 1$:

$$\begin{aligned} \sqrt{3} - 1 = x'(0) &= e^0(-c_3 \sin 0 + c_4 \cos 0) - e^0(c_3 \cos 0 + c_4 \sin 0) \\ &= c_4 - c_3. \end{aligned}$$

  Since $c_3 = 1$ this gives $c_4 = \sqrt{3}$, so the final solution is

$$x(t) = e^{-t}\left(\cos t + \sqrt{3} \sin t\right).$$

- Why did I choose the initial conditions $x(0) = 1$ and $x'(0) = \sqrt{3} - 1$? Because this makes the amplitude and phase have nice formulas:

$$x(t) = e^{-t} \cdot 2 \cos\left(t - \frac{\pi}{3}\right).$$

  Remark: The amplitude goes to zero as $t \to \infty$ because e^{-t} goes to zero. In terms of physics, we can think of this as a damped oscillator.

- In this example we worked out all of the steps, but in practice you will learn how to skip through them. If the characteristic equation has roots $\lambda_1, \lambda_2 = a \pm ib$ then you will immediately know that the general solution is

$$x(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt)).$$

We get growth or decay from the real part a and we get oscillation from the imaginary part ib.

- Here's an example with no oscillation. Textbook Problem 2.3.21:

$$y'' - 4y' + 3y = 0; y(0) = 7, y'(0) = 11.$$

I assume that the independent variable is called x. Substitute $y(x) = e^{\lambda x}$ to get characteristic equation

$$\begin{aligned} \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 1)(\lambda - 3) &= 0. \end{aligned}$$

The characteristic roots are 1 and 3 so the general solution is

$$y(x) = c_1 e^{1x} + c_2 e^{3x}.$$

To find c_1 and c_2 we first compute $y'(x) = c_1 e^x + 3c_2 e^{3x}$ to get

$$\begin{cases} y(x) &= c_1 e^{1x} + c_2 e^{3x}, \\ y'(x) &= c_1 e^{1x} + 3c_2 e^{3x}. \end{cases}$$

Substituting $y(0) = 7$ and $y'(0) = 11$ gives a system of two equations in two unknowns:

$$\begin{cases} 7 &= c_1 + c_2, \\ 11 &= c_1 + 3c_2. \end{cases}$$

Subtract these equations to get $4 = 2c_2$ so that $c_2 = 2$ and then back-substitute to obtain $c_1 = 5$. The final solution is

$$y(x) = 5e^x + 2e^{3x}.$$

- Next time we will discuss what happens when the characteristic equation has a repeated root. Preview: The equation $y'' - 4y' + 4 = 0$ has characteristic equation $\lambda^2 - 4\lambda + 4 = 0$ which factors as $(\lambda - 2)^2 = 0$. Thus there is only one characteristic root $\lambda_1 = 2$. We know that $y_1(x) = e^{2x}$ is a solution, **but we need two basic solutions!** The trick is to observe that $y_2(x) = xe^{2x}$ also works.

Wed, Feb 22

- The general story: Consider the *second order, linear, homogeneous ODE with constant coefficients*:

$$mx'' + \gamma x' + kx = 0.$$

To find the general solution we first look for basic solutions of the form $x(t) = e^{\lambda t}$. Substituting this into the equation and simplifying gives the *characteristic equation*:

$$m\lambda^2 + \gamma\lambda + k = 0,$$

which in general has two solutions:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

There are three cases.

- If $\gamma^2 - 4mk > 0$ then $\sqrt{\gamma^2 - 4mk}$ is real and there are two distinct real roots. The general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

- If $\gamma^2 - 4m < 0$ then there are two distinct complex roots, which we can write as $\lambda_1, \lambda_2 = a \pm ib$ for some real numbers a, b. Then the general solution is

$$x(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt)).$$

Note that a causes growth/decay and b causes oscillation.

- If $\gamma^2 - 4mk = 0$ then then are two equal real roots $\lambda_1 = \lambda_2 = -\gamma/2m$. In this case the general solution is

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda_1 t}.$$

- **Where did the extra solution $te^{\lambda t}$ come from?** I would like to explain this, but for now let's just check that it works in a special case. Consider the ODE $x'' + 2x' + x = 0$ with characteristic equation

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda + 1)^2 &= 0. \end{aligned}$$

This equation has a repeated root $\lambda = -1$, so I claim that the general solution is

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

We already know that e^{-t} satisfies the equation. (That's the whole point of the characteristic equation.) Let's check that $x(t) = te^{-t}$ also works. First we note that

$$\begin{aligned} x(t) &= te^{-t} \\ x'(t) &= -te^{-t} + e^{-t} = (-t + 1)e^{-t} \end{aligned}$$

$$
\begin{aligned}
x''(t) &= [(-t+1)e^{-t}]' \\
&= -(-t+1)e^{-t} - 1e^{-t} = (t-2)e^{-t}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
x'' + 2x' + x &= (t-2)e^{-t} + 2(-t+1)e^{-t} + te^{-t} \\
&= e^{-t}\left[(t-2) + 2(-t+1) + t\right] \\
&= e^{-t} \cdot 0 \\
&= 0,
\end{aligned}
$$

as desired. So it works.

- We'll see why it works later. For now we'll just memorize the rule. (When people feel ashamed about memorizing tricks without understanding, they call it an *ansatz*, which makes it sound more impressive.)
- Then we did a couple of problems from the homework.

Fri, Feb 24

- Next week:
  - Mon: Discuss Homework 3
  - Wed: Review
  - Fri: Exam
- Today I will give a preview of what comes **after** the exam. Recall that $mx'' + kx = 0$ is an undamped oscillator and $mx'' + \gamma x' + kx = 0$ is a damped oscillator. I should say that this is a *free* oscillator with no external forces acting on it.
- If an external force $f(t)$ acts on the system then the equation becomes

$$
\begin{aligned}
\text{(mass)(acceleration)} &= \text{(force)} \\
\text{(mass)(acceleration)} &= \text{(internal forces)} + \text{(external forces)} \\
\text{(mass)(acceleration)} &= \text{(stiffness and friction)} + \text{(external forces)} \\
mx''(t) &= -kx(t) - \gamma x'(t) + \text{(external forces)} \\
mx''(t) + \gamma x'(t) + kx(t) &= \text{(external forces)} \\
mx''(t) + \gamma x'(t) + kx(t) &= f(t).
\end{aligned}
$$

  The free oscillator corresponds to $f(t) = 0$, i.e., no external forces.
- First example: Maybe the spring is hanging vertically, with $x(t)$ corresponding to "up". Then the mass at the end of the spring experiences a **constant** gravitational force $-gm$:

$$
mx''(t) + \gamma x'(t) + kx(t) = -gm.
$$

To emphasize the math, let's make the constants simple:

$$x'' + x = 7.$$

Later we will see that the general solution has the form

$$x(t) = x_h(t) + x_p(t),$$

where $x_p(t)$ is **any one particular solution** and $x_h(t)$ is the general solution of the associated "homogeneous equation" $x'' + x = 0$. We are very familiar with the homogeneous solution:

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

How can we find a particular solution $x_p(t)$? We will learn general methods. The first is the *method of undetermined coefficients*, which really just means "make an educated guess". When the right hand side is a constant, the educated guess is a constant. Suppose that $x_p(t) = c$ for some constant c, so $x_p'(t) = 0$ and $x_p''(t) = 0$. If $x_p(t)$ satisfies the equation then we have

$$\begin{aligned} x_p''(t) + x_p(t) &= 7 \\ 0 + c &= 7 \\ c &= 7. \end{aligned}$$

And, indeed, we can check that $x_p(t) = 7$ is a solution. So the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1 \cos t + c_2 \sin t + 7.$$

This just describes oscillations around the new equilibrium $x = 7$.

- Similarly, a hanging spring just oscillates around a new equilibrium, which is lower than it would be without gravity.

- Way more interesting: Periodic forcing. Suppose now that the spring is subject to a periodic force $f(t) = \cos(\omega t)$, so that

  $$x'' + x = \cos(\omega t).$$

  Again, the solution is $x(t) = x_h(t) + x_p(t) = c_1 \cos t + c_2 \sin t + x_p(t)$. There are a few different ways to find a particular solution $x_p(t)$:

  - method of undetermined coefficients,
  - variation of parameters,
  - Laplace transforms.

  We'll discuss this later. For lack of time I'll just tell you the solution:

  $$x(t) = c_1 \cos t + c_2 \sin t + \left(\frac{1}{1 - \omega^2} \right) \cos(\omega t).$$

  Note that the solution explodes when ω approaches 1. This is called *resonance*. The *natural frequency* of the unforced oscillator $x'' + x = 0$ is 1. If we shake the spring at a frequency close to 1 then the amplitude will grow. If ω is too close to 1 then the spring will break. Engineers must learn to avoid this.

Mon, Feb 27

- We discussed the Homework 3 solutions. Please read the solutions:

  `https://www.math.miami.edu/~armstrong/311sp23/311sp23hw3sol.pdf`

March

Wed, Mar 1

- We discussed the practice problems for the exam. Please read the solutions:

  `https://www.math.miami.edu/~armstrong/311sp23/311sp23exam1practice_solutions.pdf`

Fri, Mar 3

Exam day.

Mon, Mar 6

- Moving on. The simplest kind of equation that we don't know how to solve is

  $$ax''(t) + bx'(t) + cx(t) = f(t),$$

  for some constants a, b, c and some nonzero function $f(t)$. We will learn at least three methods to solve this equation.

- Let me take this opportunity to introduce some general theory. An *operator* L is a rule that sends functions to functions:

  $$L : \text{functions} \to \text{functions}$$

  Notation: The operator L sends the function $y(x)$ to $L[y(x)]$. We call L a *linear operator* if it satisfies the following two properties:

  - for any function $y(x)$ and constant C we have

    $$L[Cy(x)] = CL[y(x)].$$

  - for any two functions $y_1(x)$ and $y_2(x)$ we have

    $$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)].$$

- You have already been working with linear operators without realizing it. For example, the *derivative operator* $L[y(x)] = y'(x)$ is linear. Indeed, you learned in Calc I that

  $$L[Cy(x)] = (Cy(x))' = Cy'(x) = CL[y(x)]$$

and

$$L[y_1(x) + y_2(x)] = (y_1(x) + y_2(x))' = y_1'(x) + y_2'(x) = L[y_1(x)] + L[y_2(x)].$$

For the same reasons, the *n-th derivative operator* $L[y(x)] = y^{(n)}(x)$ is linear.

- For any constants a, b, c the operator $L[y(x)] = ay''(x) + by'(x) + cy(x)$ is linear.
- For any function $P(x)$ the operator $L[y(x)] = P(x)y(x)$ that multiplies $y(x)$ by $P(x)$ linear. Indeed, we have

$$L[Cy(x)] = P(x)Cy(x) = CP(x)y(x) = CL[y(x)]$$

and

$$\begin{aligned} L[y_1(x) + y_2(x)] &= P(x)(y_1(x) + y_2(x)) \\ &= P(x)y_1(x) + P(x)y_2(x) \\ &= L[y_1(x)] + L[y_2(x)]. \end{aligned}$$

- Putting all of these together, the *most general linear operator* has the form

$$L[y(x)] = P_0(x)y^{(n)}(x) + P_1(x)y^{(n-1)}(x) + \cdots + P_{n-1}(x)y'(x) + P_n(x)y(x)$$

for some functions $P_0(x), P_1(x), \ldots, P_n(x)$.

- Examples of **non-linear operators**:

$$\begin{aligned} L[y(x)] &= 5, \\ L[y(x)] &= y(x)^2, \\ L[y(x)] &= y'(x)y(x). \end{aligned}$$

Constant terms and **products** of derivatives of $y(x)$ are not linear.

- A *linear ODE* has the form

$$L[y(x)] = f(x)$$

for some linear operator L and some function $f(x)$. We have good methods to solve linear ODEs. Non-linear ODEs are impossible to solve exactly, except for some small examples such as $y'(x) - y(x)^2 = 0$, which you saw on exam. The general method to deal with non-linear equations is to "linearize" them. So the theory of linear ODEs is of central importance.

- **The general method.** To solve the linear equation $L[y(x)] = f(x)$.

  (1) Let $y_c(x)$ be the general solution of the associated *homogeneous equation*

$$L[y(x)] = 0.$$

  The letter c stands for "complementary", which is the word our textbook uses for homogeneous solutions. I don't like that word, but oh well.

(2) Let $y_p(x)$ be any one particular solution of $L[y(x)] = f(x)$.

(3) Then the general solution of $L[y(x)] = f(x)$ is

$$y(x) = \underbrace{y_c(x)}_{\text{general homogeneous solution}} + \underbrace{y_p(x)}_{\text{one particular solution}}.$$

- Let's see an example. On the exam you solved the equation $y'(x) = x + y(x)$ using the method of integrating factors. We can rewrite this equation as $L[y(x)] = x$, for the linear operator $L[y(x)] = y'(x) - y(x)$.

  (1) The associated homogeneous equation is $L[y(x)] = 0$, i.e., $y'(x) - y(x) = 0$, which has general solution
  $$y_c(x) = Ce^x.$$
  as you well know.

  (2) Now we just need to find one particular solution of $L[y(x)] = x$. The quickest method is an inspired guess called the *method of undetermined coefficients*. Since the right hand side is a polynomial of degree 1, we look for a polynomial solution of degree 1: $y_p(x) = Ax + B$. And this does work:
  $$\begin{aligned} y_p'(x) - y_p(x) &= x \\ (Ax+B)' - (Ax+B) &= x \\ A - (Ax+B) &= x \\ -Ax + (A-B) &= 1x + 0. \end{aligned}$$
  Thus we can take $A = -1$ and $B = -1$ to get particular solution $y_p(x) = -x - 1$.

  (3) The general solution of $L[y(x)] = x$ is
  $$y(x) = y_c(x) + y_p(x) = Ce^x + (-x-1).$$

- For first order linear equations $y'(x) + P(x)y(x) = f(x)$ (such as the previous example) we already had the method of integrating factors. But for second order equations $y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$ we do not yet have any method.

- Example. Consider the second order linear equation $y''(x) + y(x) = x$.

  (1) The associated homogeneous equation $y''(x) + y(x) = 0$ has general solution
  $$y_c(x) = A\cos x + B\sin x,$$
  as you well know.

  (2) Now we need a particular solution $y_p(x)$. Since the right side x is a polynomial of degree 1, the method of undetermined coefficients suggest $y_p(x) = Cx + D$. And this does work:
  $$y_p''(x) + y_p(x) = x$$

$$\begin{aligned}(Cx+D)''+(Cx+D)&=x\\0+Cx+D&=x.\end{aligned}$$

Taking $C=1$ and $D=0$ gives the solution $y_p(x)=x$.

(3) Hence the general solution of $y''(x)+y(x)=x$ is

$$y(x)=y_c(x)+y_p(x)=A\cos x+B\sin x+x.$$

That's something new.

Wed, Mar 8

- Review: We only know how to solve *linear* ODEs. A linear ODE has the form

$$L[y(x)]=f(x),$$

where L is a linear operator. A linear operator must satisfy $L[Cy(x)]=CL[y(x)]$ and $L[y_1(x)+y_2(x)]=L[y_1(x)]+L[y_2(x)]$. The most general linear operator looks like

$$L[y(x)]=P_0(x)y(x)+P_1(x)y'(x)+\cdots+P_n(x)y^{(n)}(x),$$

for some functions $P_0(x),\ldots,P_n(x)$.

- The general solution of a linear equation $L[y(x)]=f(x)$ is $y(x)=y_c(x)+y_p(x)$, where $y_c(x)$ is the general solution of the *homogeneous* equation $L[y(x)]=0$ and $y_p(x)$ is any one particular solution of $L[y(x)]=f(x)$.

- Example from last time: $y'-y=x$. The general solution of the homogeneous equation $y'-y=0$ is $y(x)=Ce^x$. To find a particular solution to $y'-y=x$ we **guess** that $y_p(x)=Ax+B$ is a solution for some constants A and B. (This guess is called the "method of undetermined coefficients".) Substituting this guess into the equation gives $A=-1$ and $B=-1$, so the guess worked. The general solution is

$$y(x)=y_c(x)+y_p(x)=Ce^x+(-x-1).$$

- The "method of undetermined coefficients" is not really a method; it is just some rules for guessing, and it only works for certain functions $f(x)$. You can find these rules on pages 155 and 156 of the textbook. I won't make you memorize these rules. If I want you to use the method I will just tell you what to guess.

- Example from page 152:

$$y''+4y=3x^3.$$

The homogeneous equation $y''+4y=0$ has general solution $y(x)=c_1\cos(2x)+c_2\sin(2x)$, obtained using previous methods. To find one particular solution, we note that the right hand side $3x^3$ is a polynomial of degree 3, so we guess a polynomial of degree 3:

$$\text{Guess:}\quad y_p(x)=Ax^3+Bx^2+Cx+D.$$

Substituting this guess into $y'' + 4y = 3x^2$ allows us to solve for A, B, C, D. The solution is $A = 3/4$, $B = 0$, $C = -9/8$ and $D = 0$, so that $y_p(x) = \frac{3}{4}x^3 - \frac{9}{8}x$ is a solution. The general solution of $y'' + 4y = 3x^2$ is

$$y(x) = y_c(x) + y_p(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{3}{4}x^3 - \frac{9}{8}x.$$

- The last example was just a textbook problem. Now an actually interesting example:

  $$x''(t) + x(t) = \cos(\omega t).$$

  The homogeneous equation is an undamped oscillator: $x''(t) + x(t) = 0$ with solution

  $$x_c(t) = c_1 \cos t + c_2 \sin t.$$

  In the non-homogeneous equation $x''(t) + x(t) = \cos(\omega t)$, we are applying a periodic external force with frequency ω. Idea: The system **wants** to vibrate at its "natural frequency" 1. We are trying to force it to vibrate at frequency ω. What will happen?

  The right hand side has the form $A\cos(\omega t) + B\sin(\omega t)$, so the method of undetermined coefficients guesses a solution of the same form:

  $$x_p(t) = A\cos(\omega t) + B\sin(\omega t).$$

  We compute

  $$\begin{aligned} x_p'(t) &= -A\omega \sin(\omega t) + B\omega \cos(\omega t), \\ x_p''(t) &= -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t). \end{aligned}$$

  Note that $x_p''(x) = -\omega^2 x_p(t)$. That's a lucky simplification. Plug this guess into $x''(t) + x(t) = \cos(\omega t)$ to get

  $$\begin{aligned} x_p''(t) + x_p(t) &= \cos(\omega t) \\ -\omega^2 x_p(t) + x_p(t) &= \cos(\omega t) \\ x_p(t)(1 - \omega^2) &= \cos(\omega t) \\ x_p(t) &= \frac{1}{1 - \omega^2} \cdot \cos(\omega t). \end{aligned}$$

  The lucky simplification allowed us to jump right to the solution without solving for A and B directly. You can also do it the slow way.

  The general solution of $x''(t) + x(t) = \cos(\omega t)$ is

  $$x(t) = c_1 \cos t + c_2 \sin t + \frac{1}{1 - \omega^2} \cdot \cos(\omega t).$$

  If the forcing frequency ω is very far from the natural frequency 1 then $1/(1 - \omega^2)$ is close to 0 and the solution is close to the unforced solution:

  $$x(t) \approx c_1 \cos t + c_2 \sin t.$$

  If the forcing frequency ω is close to the natural frequency 1 then $1/(1 - \omega^2)$ is huge and the forcing term dominates. If ω is too close to 1 then the system will explode. This is called "resonance".

Fri, Mar 10

- Consider a linear ODE:
$$L[y(x)] = f(x).$$
The general solution is $y(x) = y_c(x) + y_p(x)$ where $y_p(x)$ is **any one particular solution** and $y_c(x)$ is the **general solution of the associated homogeneous equation** $L[y(x)] = 0$.

- We already know how to solve the homogeneous equation $L[y(x)] = 0$. (At least when L has constant coefficients; non-constant coefficients lead to new kinds of functions such as Bessel functions.) Thus our new problem is to find one particular solution $y_p(x)$.

- We will learn three methods:

  (1) **Undetermined Coefficients.** If $f(x)$ has a simple form then we can make an educated guess for $y_p(x)$ then solve a system of linear equations to get the parameters. Upside: When it works it is often fast. Downside: Guessing involves memorization and sometimes it doesn't work.

  (2) **Variation of Parameters.** See below. This method always works but it requires us to solve some possibly difficult integrals. Fine for computers, but bad for humans.

  (3) **Laplace Transforms.** This is a powerful method for hand computation. With practice it is more powerful than the method of undetermined coefficients and the computations are not so bad. Downside: You will need a table to look up the transforms. (There is no free lunch.)

- **Method of Undetermined Coefficients.** Consider $L[y(x)] = f(x)$ where L has constant coefficients. Here are the good guesses:

  - If $f(x)$ is a polynomial of degree n let $y_p(x)$ be a polynomial of degree n.
  - If $f(x) = A\cos(\omega x) + B\sin(\omega x)$ let $y_p(x) = C\cos(\omega x) + D\cos(\omega x)$.
  - If $f(x) = e^{rx}$ let $y_p(x) = Ce^{rx}$.

  "Rule 2" on page 155 summarizes these guesses.

- Example: $y'' + 16y = e^{3x}$. The general solution to $y'' + 16y = 0$ is
$$y_c(x) = c_1\cos(4x) + c_2\sin(4x).$$
To find a particular solution we guess $y_p(x) = Ce^{3x}$ and substitute to get
$$\begin{aligned} y_p''(x) + 16y_p(x) &= e^{3x} \\ 9Ce^{3x} + 16Ce^{3x} &= e^{3x} \\ 25Ce^{3x} &= e^{ex} \\ 25C &= 1 \\ C &= 1/25. \end{aligned}$$

Hence $y_p(x) = \frac{1}{25}e^{3x}$ is a solution. The guess worked because all the derivatives of e^{3x} have the form (constant)e^{3x}, so e^{3x} is a common factor of both sides. Hence the general solution of $y'' + 16y = e^{3x}$ is

$$y(x) = y_c(x) + y_p(x) = c_1 \cos(4x) + c_2 \sin(4x) + \frac{1}{25}e^{3x}.$$

There are two parameters because the ODE has second order.

- Example: $x''(t) + 9x(t) = 80\cos(5t)$. The general solution of $x''(t) + 9x(t) = 0$ is

$$x_c(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

Since the right hand side has the form $\cos(5t)$ we make the guess

$$x_p(t) = A\cos(5t) + B\sin(5t).$$

We know the guess will work because every derivative of $x_p(t)$ still has the same form. (In fact, since $x'(t)$ does not appear on the left, we could just take $x_p(t) = A\cos(5t)$, but I'll show the general method.) Compute

$$\begin{aligned} x_p'(t) &= -5A\sin(5t) + 5B\cos(5t), \\ x_p''(t) &= -25A\cos(5t) - 25B\sin(5t), \end{aligned}$$

then substitute:

$$\begin{aligned} 80\cos(5t) &= x_p''(t) + 9x_p(t) \\ &= (-25A\cos(5t) - 25B\sin(5t)) + 9(A\cos(5t) + B\sin(5t)) \\ &= (-16A)\cos(5t) + (-16B)\sin(5t). \end{aligned}$$

Comparing coefficients gives $-16A = 80$ and $-16B = 0$, hence $A = -5$ and $B = 0$. The general solution of $x''(t) + 9x(t) = 30\cos(5t)$ is

$$x(t) = x_c(t) + x_p(t) = c_1 \cos(3t) + c_2 \sin(3t) - 5\cos(5t).$$

Now suppose initial conditions $x(0) = 0$ and $x'(0) = 0$. Substituting $x(0) = 0$ gives $c_1 = 5$ and substituting $x'(0) = 0$ gives (after a bit of work) $c_2 = 0$, hence the solution is

$$x(t) = x_c(t) + x_p(t) = 5\cos(3t) - 5\cos(5t).$$

Interpretation: A spring with natural frequency 3 is being forced to vibrate with frequency 5. The solution is a superposition of these two frequencies.

- **Intro to Variation of Parameters.** To solve $L[y(x)] = f(x)$, first find the general homogeneous solution

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x).$$

(For simplicity, we assume that L has second order.) Then we can **always** find a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

for some functions $u_1(x)$ and $u_2(x)$. (That is, we let the parameters vary.) In fact, there are **general formulas** for $u_1(x)$ and $u_2(x)$:

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)f(x)}{W(x)}\,dx, \\ u_2(x) &= +\int \frac{y_1(x)f(x)}{W(x)}\,dx, \end{aligned}$$

where $W(x)$ is the "Wronskian function"

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

So the general solution of $L[y(x)] = f(x)$ is

$$y(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx.$$

- Example: The basic solutions of $x''(t)+9x(t) = 0$ are $x_1(t) = \cos(3t)$ and $x_2(t) = \sin(3t)$. The Wronskian function is

$$\begin{aligned} W(t) &= x_1(t)x_2'(t) - x_1'(t)x_2(t) \\ &= 3\cos^2(3t) + 3\sin^2(3t) \\ &= 3. \end{aligned}$$

Hence the general solution of $x''(t) + 9x(t) = 80\sin(5t)$ is

$$x(t) = -\cos(3t)\int \frac{\sin(3t)\cdot 80\cos(5t)}{3}\,dt + \sin(3t)\int \frac{\cos(3t)\cdot 80\cos(5t)}{3}\,dt.$$

The parameters c_1 and c_2 will come from the two integrals. Is this equivalent to our previous solution? Yes, but it's pretty hard to see that. This method is good for computers; not so good for humans.

- Because this method is not good for humans, I won't dwell on it for long. We will quickly move on to the method of **Laplace transforms**, which is good for humans.

Mon, Mar 20

- Welcome back from spring break. (Does this need an exclamation point?)
- Today: One example of undetermined coefficients; then some variation of parameters.

- Solve $x''(t) + 1600x(t) = -336\sin(44t)$ with $x(0) = 0$ and $x'(0) = 84$. (I reverse-engineered this problem so the solution look nice.)
  - The homogeneous equation $x''(t) + 1600x(t) = 0$ has general solution

$$x_c(t) = c_1 \cos(40t) + c_2 \sin(40t).$$

  - To find a non-homogeneous solution we guess

$$x_p(t) = A\cos(44t) + B\sin(44t).$$

    Substitute this into $x''(t) + 1600x(t) = -336\sin(44t)$ and simplify to obtain $A = 0$ and $B = 1$, hence

$$x_p(t) = \sin(44t).$$

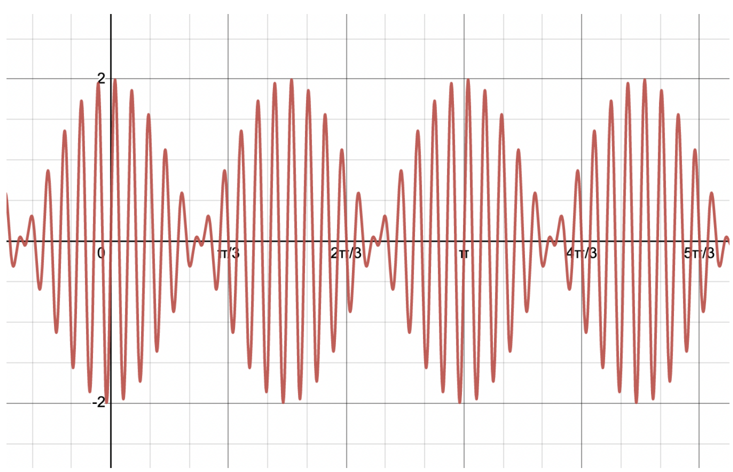
    (This worked out because $-44^2 + 1600 = -336$.)
  - Hence the general solution is

$$x(t) = x_c(t) + x_p(t) = c_1 \cos(40t) + c_2 \sin(40t) + \sin(44t).$$

  - Substitute the initial conditions $x(0) = 0$ and $x'(0) = 84$ to get $c_1 = 0$ and $c_2 = 1$. (This works out because $84 = 40 + 44$.) Hence the solution is

$$x(t) = \sin(40t) + \sin(44t).$$

- **Interpretation of the solution.** A simple oscillator with natural frequency 40 is subjected to a periodic force with frequency 44. Here is the graph of the solution:



What's going on here? To understand this we must consider the trig identities:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v,$$
$$\sin(u - v) = \sin u \cos v - \cos u \sin v,$$

$$\sin(u+v) + \sin(u-v) = 2\sin u \cos v.$$

Substitute $\alpha = u + v$ and $\beta = u - v$ to get

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

This identity explains our solution. Substitute $\alpha = 44t$ and $\beta = 40t$ to get

$$x(t) = \sin(44t) + \sin(40t) = 2\sin(42t)\cos(2t).$$

Since 42 is much bigger than 2, this looks like a sine wave oscillating quickly between the values $2\cos(2t)$ and $-2\cos(2t)$. Example: If two musical notes of frequencies 400 Hz and 440 Hz are played together, you will hear a note with frequency 420 Hz switching on and off with frequency 20 Hz. This is the phenomenon of "beats".

- The method of **undetermined coefficients** is just an educated guess. It only works when the non-homogeneous term has a nice form.
- The method of **variation of coefficients** always works, but the computations are more difficult.
- Simple Example: Solve $y'(x) + y(x) = e^x$ using variation of parameters.
  - The homogeneous equation $y'(x) + y(x) = 0$ has general solution

    $$y_c(x) = Ce^{-x}.$$

  - To find a particular solution, turn the parameter into a function:[1]

    $$y_p(x) = u(x)e^{-x}.$$

  - Substitute into the equation and use the product rule to get

    $$\begin{aligned} y_p''(x) + y_p(x) &= e^x \\ [u'(x)e^{-x} - \cancel{u'(x)e^{-x}}] + \cancel{u(x)e^{-x}} &= e^x \\ u'(x)e^{-x} &= e^x \\ u'(x) &= e^{2x} \\ u(x) &= \int e^{2x}\,dx \\ u(x) &= \frac{1}{2}e^{2x}. \end{aligned}$$

    (Any antiderivative of e^{2x} is fine.) Hence we obtain

    $$y_p(x) = u(x)e^{-x} = \frac{1}{2}e^{2x}e^{-x} = \frac{1}{2}e^x,$$

    and the general solution of the linear equation $y'(x) + y(x) = e^x$ is

    $$y(x) = y_c(x) + y_p(x) = Ce^{-x} + \frac{1}{2}e^x.$$

[1] I would call the function $C(x)$, but the notation $u(x)$ seems standard.

- Remark: The method of variation of parameters for first order equations is secretly the same as the method of integrating factors. For second order equations, variation of parameters gives us something new.

- Solve $x''(t) + 1600x(t) = -336\sin(44t)$ using variation of parameters:

  - The general homogeneous solution is $x_c(t) = c_1\cos(40t) + c_2\sin(40t)$.
  - To find a particular solution, turn the parameters into functions:

$$x_p(t) = u_1(t)\cos(40t) + u_2(t)\sin(40t).$$

  - The rest **you do not want to do by hand!** After substituting and making the simplifying assumption that $u_1'(t)\cos(40t) + u_2'(t)\sin(40t) = 0$ we will obtain

$$-40u_1'(t)\sin(40t) + 40u_2'(t)\cos(40t) = -336\sin(44t).$$

    Solve these two simultaneous equations to obtain $u_1'(t)$ and $u_2'(t)$ then integrate to get $u_1(t)$ and $u_2(t)$. The answer will look like a big mess.
  - But we already found the particular solution $x_p(t) = \sin(44t)$ using the method of undetermined coefficients, which is a much better method to solve this problem. (Variation of parameters is usually used when the method of undetermined coefficients doesn't work.)

- Next time I'll show you a toy example of a second order equation for which the method of variation of parameters can be done by hand.

Wed, Mar 22

- **Variation of Parameters.** Consider a general second order linear equation:

$$\begin{aligned} L[y(x)] &= f(x) \\ y''(x) + P(x)y'(x) + Q(x)q(x) &= f(x), \end{aligned}$$

  for some functions $P(x), Q(x), f(x)$. Suppose the general solution of $L[y(x)] = 0$ is

$$y_c(x) = c_1y_1(x) + c_2y_2(x).$$

  Then we can find a particular solution of $L[y(x)] = f(x)$ of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

  for some functions $u_1(x), u_2(x)$. Our goal is to solve for $u_1(x)$ and $u_2(x)$. In order to do this **we will need two equations**. Substituting $y_p(x)$ into $L[y(x)] = f(x)$ only gives one equation, therefore we are free to impose any second equation that we choose. It is most convenient to assume

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0.$$

(This is just a good trick.) Then we have

$$\begin{aligned} y_p'(x) &= [u_1y_1 + u_2y_2]' \\ &= u_1'y_1 + u_1y' + u_2'y_2 + u_2'y_2 \\ &= u_1y_1' + u_2y_2' + (u_1'y_1 + u_2'y_2) \\ &= u_1y_1' + u_2y_2' + 0 && \text{trick} \\ &= u_1y_1' + u_2y_2'' \end{aligned}$$

and

$$\begin{aligned} y_p''(x) &= [u_1y' + u_2y_2']' \\ &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \\ &= (u_1'y_1' + u_2'y_2') + (u_1y_1'' + u_2y_2''). \end{aligned}$$

Recall that each of $y_1(x)$ and $y_2(x)$ is a solution of $y''(x) + P(x)y'(x) + Q(x) = 0$. After substituting our expressions for $y_p''(x)$ and $y_p'(x)$ into the equation $y_p'' + Py_p' + Qy_p = f$ and simplifying, we obtain

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x).$$

Thus we have two equations for the two unknown functions u_1' and u_2':

$$\begin{cases} u_1'y_1 & + & u_2'y_2 & = & 0, \\ u_1'y_1' & + & u_2'y_2' & = & f. \end{cases}$$

As long as y_1 and y_2 are independent[2] then this system of equations has a unique solution for u_1' and u_2':

$$\begin{cases} u_1' & = & y_2f/(y_1y_2' - y_1'y_2), \\ u_2' & = & -y_1f/(y_1y_2' - y_1'y_2). \end{cases}$$

(Don't memorize these formulas. It's easier just to solve the equations in each particular example.) Then we simply (maybe not so simply) integrate to get u_1 and u_2.

- This method is quite general but the computations can be difficult. Here is a very simple example:

$$y''(x) - 3y'(x) + 2y(x) = 1,$$

so $P(x) = -3$, $Q(x) = 2$ and $f(x) = 1$. Since $\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$, the general homogeneous solution is

$$y_c(x) = c_1e^x + c_2e^{2x}.$$

Hence there exists a particular solution of the form

$$y_p(x) = u_1(x)e^x + u_2(x)e^{2x},$$

[2]Technically, we require $y_1y_2' - y_1'y_2 \neq 0$. This is guaranteed when $c_1y_1 + c_2y_2$ is the **general** solution of the homogeneous equation $L[y(x)] = 0$.

where the functions $u_1(x)$ and $u_2(x)$ satisfy the equations

$$\begin{cases} u_1' y_1 & + & u_2' y_2 & = & 0, \\ u_1' y_1' & + & u_2' y_2' & = & f. \end{cases}$$

With $y_1 = e^x$, $y_2 = e^{2x}$ and $f = 1$, these become

$$\begin{cases} u_1' e^x & + & u_2' e^{2x} & = & 0, \\ u_1' e^x & + & 2u_2' e^{2x} & = & 1. \end{cases}$$

Subtracting the equations gives $u_2' e^{2x} = 1$, hence $u_2' = e^{-2x}$ and $u_2 = -e^{2x}/2$. Then back-substituting gives $u_1' e^x = -u_2' e^{2x} = -e^{-2x} e^{2x} = -1$, hence $u_1' = -e^{-x}$ and $u_1 = e^{-x}$. (Since we are looking for one particular solution we can pick any antiderivatives of u_1' and u_2'.) Finally, we obtain a particular solution:

$$y_p(x) = u_1(x)e^x + u_2(x)e^{2x} = e^{-x}e^x - \frac{1}{2}e^{-2x}e^{2x} = 1 - \frac{1}{2} = \frac{1}{2}.$$

- That solution is so simple, there must have been an easier way to get it. Indeed, since $f(x) = 1$ is constant, the method of undetermined coefficients tells us to guess $y_p(x) = A$ constant. Then $y_p'(x) = 0$ and $y_p''(x) = 0$ and substituting gives

$$\begin{aligned} y_p''(x) - 3y_p'(x) + 2y_p(x) &= 1 \\ 0 - 3 \cdot 0 + 2 \cdot A &= 1 \\ A &= 1/2. \end{aligned}$$

  The method of variation of coefficients is not that valuable when we already have a good guess; it is valuable in cases when we **don't** have a good guess, i.e., when the method of undetermined coefficients fails. But those cases are generally too complicated to do by hand, and you won't see them on an exam.

- **Laplace Transforms.** Finally. I have teased the method of Laplace transforms and now we begin to discuss it. This is the content of Chapter 4.

- The *Laplace transform* is a linear operator $\mathscr{L}$ defined as follows:

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$$

  This looks bad but it is quite useful and convenient to work with. The integral goes back to Euler and Laplace in the late 1700s but the general method was developed by practicing engineers such as Oliver Heaviside in the early 1900s.

- We will spend several lectures learning this method. Today I want to go far enough to show you one example. I'll skip some details and we'll go over them next time.

- First we note that

$$\mathscr{L}[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt = \cdots \text{details} \cdots = \frac{1}{s-a}.$$

Then we note that

$$\begin{aligned}
\mathscr{L}[f'(t)] &= \int_0^\infty e^{-st} f'(t)\, dt = \cdots \text{details} \cdots = s\mathscr{L}[f(t)] - f(0) \\
\mathscr{L}[f''(t)] &= \int_0^\infty e^{-st} f''(t)\, dt = \cdots \text{details} \cdots = s^2\mathscr{L}[f(t)] - sf(0) - f'(0).
\end{aligned}$$

- Now we know enough to solve our first problem. Consider the equation

$$x''(t) - 3x'(t) + 2x(t) = 1 \quad \text{with initial conditions } x(0) = x'(0) = 0.$$

Let $X(s) = \mathscr{L}[x(t)]$ denote the Laplace transform of $x(t)$. Applying $\mathscr{L}$ gives

$$\begin{aligned}
\mathscr{L}[x''(t) - 3x'(t) + 2x(t)] &= \mathscr{L}[1] \\
\mathscr{L}[x''(t)] - 3\mathscr{L}[x'(t)] + 2\mathscr{L}[x(t)] &= \mathscr{L}[e^{0t}] \\
s^2X(s) - 3sX(s) + 2X(s) &= \frac{1}{s} \\
X(s) &= \frac{1}{s(s^2 - 3s + 2)} \\
&= \frac{1}{s(s-1)(s-2)} \\
&= \frac{1}{2}\cdot\frac{1}{s} - 1\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-2}. \quad \text{partial fractions}
\end{aligned}$$

We recognize these summands as Laplace transforms:

$$\mathscr{L}[1] = \frac{1}{s}, \quad \mathscr{L}[e^t] = \frac{1}{s-1}, \quad \mathscr{L}[e^{2t}] = \frac{1}{s-2}.$$

Hence we obtain

$$\begin{aligned}
X(s) &= \frac{1}{2}\cdot\frac{1}{s} - 1\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-2} \\
\mathscr{L}[x(t)] &= \frac{1}{2}\mathscr{L}[1] - \mathscr{L}[e^t] + \frac{1}{2}\mathscr{L}[e^{2t}] \\
\mathscr{L}[x(t)] &= \mathscr{L}\left[\frac{1}{2} - e^t + \frac{1}{2}e^{2t}\right] \\
x(t) &= \frac{1}{2} - e^t + \frac{1}{2}e^{2t}.
\end{aligned}$$

by applying the inverse transform $\mathscr{L}^{-1}$ to both sides. That's a pretty typical calculation. It's slower than the easy guess $x(t) =$ constant, but it's faster than variation of parameters.

Fri, Mar 24

- The *Laplace transform* is a certain linear operator with miraculous properties. Given any (reasonably nice) function $f(t)$, the Laplace transform is defined by

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$$

  - Note that $\mathscr{L}[f(t)]$ is a function of s. It is common to write $F(s) = \mathscr{L}[f(t)]$.
  - The letters s is not important. However, we usually think of t as time since the integral goes from $t = 0$ to $t = \infty$. (That is, we don't consider $t < 0$.)

- "The calculus" is really just a bag of tricks for computing derivatives, developed by Newton and Leibniz in the mid 1600s. "The method of Laplace transforms" is a bag of tricks for solving ODEs, developed by Heaviside in the late 1800s. So we could also call it "the Heaviside calculus".

- The Newton-Leibniz calculus is based on a few rules, such as $(c_1 f_1(t) + c_2 f_2(t))' = c_1 f_1'(t) + c_2 f_2'(t)$ and $(x^n)' = nx^{n-1}$ and $(f(t)g(t))' = f'(t)g(t) + f(t)g'(t)$. Our goal is to find the basic rules of Heaviside calculus.

- The most important rule is that $\mathscr{L}$ is linear, which follows from the fact that integration is linear:

$$\begin{aligned}\mathscr{L}[c_1 f_1(t) + c_2 f_2(t)] &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\, dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt \\ &= c_1 \mathscr{L}[f_1(t)] + c_2 \mathscr{L}[f_2(t)].\end{aligned}$$

- The second most important rule is $\mathscr{L}[e^{at}] = 1/(s-a)$. Proof:

$$\begin{aligned}\mathscr{L}[e^{at}] &= \int_0^\infty e^{-st} e^{at}\, dt \\ &= \int_0^\infty e^{(a-s)t}\, dt \\ &= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty \\ &= \frac{1}{a-s}[0-1] \\ &= \frac{1}{s-a}.\end{aligned}$$

  This includes the special case $\mathscr{L}[1] = \mathscr{L}[e^{0t}] = 1/(s-0) = 1/s$.

- The third most important rule is $\mathscr{L}[f'(t)] = s\mathscr{L}[f(t)] - f(0)$. Proof:

$$\mathscr{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$$
\begin{aligned}
&= \int u dv && u = e^{-st} \text{ and } v = f(t) \\
&= uv - \int v du \\
&= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\, dt \\
&= e^{-\infty} f(\infty) - f(0) + s \int e^{-st} f(t)\, dt \\
&= 0 - f(0) + s\mathscr{L}[f(t)].
\end{aligned}
$$

In this calculation we assumed that $s > 0$ so $e^{-st} \to 0$ as $t \to \infty$ and we assumed that $f(t)$ grows slower than an exponential so that $e^{-st} f(t) \to 0$ as $t \to \infty$.

- These three rules are already enough to solve some simple ODEs. Example:

$$x'(t) - x(t) = 0.$$

Let $X(s) = \mathscr{L}[x(t)]$. Apply $\mathscr{L}$ to both sides:

$$
\begin{aligned}
\mathscr{L}[x'(t) - x(t)] &= \mathscr{L}[0] \\
\mathscr{L}[x'(t)] - \mathscr{L}[x(t)] &= 0 && (\mathscr{L}[0] = 0) \\
sX(s) - x(0) - X(s) &= 0 \\
(s-1)X(s) &= x(0) \\
X(s) &= x(0)\frac{1}{s-1} \\
x(t) &= \mathscr{L}^{-1}\left[x(0)\frac{1}{s-1}\right] \\
&= x(0)\mathscr{L}^{-1}\left[\frac{1}{s-1}\right] \\
&= x(0)e^t.
\end{aligned}
$$

Remark: Here we assumed that the inverse transform $\mathscr{L}^{-1}$ exists. This is true but it is very difficult to calculate from scratch. Instead we will rely on reverse table lookup. In this case we recognized $1/(s-1)$ as the transform of e^t.

- Next rule: If $\mathscr{L}[f(t)] = F(s)$ then $\mathscr{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$. Proof: Let $g(t) = f'(t)$. Then

$$
\begin{aligned}
\mathscr{L}[f''(t)] &= \mathscr{L}[g'(t)] \\
&= s\mathscr{L}[g(t)] - g(0) \\
&= s\mathscr{L}[f'(t)] - f'(0) \\
&= s\left(sF(s) - sf(0)\right) - f'(0) \\
&= s^2 F(s) - sf(0) - f'(0).
\end{aligned}
$$

- Example: $x''(t) - 3x(t) + 2x(t) = 1$. Let $X(s) = \mathscr{L}[x(t)]$ and apply $\mathscr{L}$:

$$\begin{aligned} \mathscr{L}[x''(t) - 3x(t) + 2x(t)] &= \mathscr{L}[1] \\ \mathscr{L}[x''(t)] - 3\mathscr{L}[x(t)] + 2\mathscr{L}[x(t)] &= 1/s \\ s^2X(s) - sx(0) - x'(0) - 3(sX(s) - x(0)) + 2X(s) &= 1/s. \end{aligned}$$

  For simplicity, assume $x(0) = x'(0) = 0$, so

$$\begin{aligned} s^2X(s) - 3(sX(s) + 2X(s) &= 1/s \\ (s^2 - 3s + 2)X(s) &= 1/s \\ X(s) &= \frac{1}{s(s^2 - 3s + 2)} \\ X(s) &= \frac{1}{s(s-1)(s-2)} \\ x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s(s-1)(s-2)}\right]. \end{aligned}$$

  To use reverse table lookup we must first simplify the right hand side using partial fractions:

$$\begin{aligned} \frac{1}{s(s-1)(s-2)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \\ \frac{1}{s(s-1)(s-2)} &= \frac{A(s-1)(s-2) + Bs(s-2) + Cs(s-1)}{s(s-1)(s-2)} \\ 1 &= A(s-1)(s-2) + Bs(s-2) + Cs(s-1). \end{aligned}$$

  Put $s = 0$ to get $1 = 2A$, put $s = 1$ to get $1 = -B$ and put $s = 2$ to get $1 = 2C$. Hence

$$\begin{aligned} \frac{1}{s(s-1)(s-2)} &= \frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2} \\ \mathscr{L}^{-1}\left[\frac{1}{s(s-1)(s-2)}\right] &= \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s-2}\right] \\ &= \frac{1}{2}\cdot 1 - e^{1t} + \frac{1}{2}e^{2t}. \end{aligned}$$

- The "Heaviside coverup method" is a shortcut for partial fractions. The coefficient of $1/(s-a)$ in the expansion of $\frac{1}{(s-a)g(t)}$ (where $g(t)$ is a polynomial not having a as a root) is $1/g(a)$. Check that it works in the previous example.

- Here's one we couldn't do before: $x'(t) - x(t) = e^t$. The method of undetermined coefficients guesses $x_p(t) = Ae^t$, but then substitution gives $0 = e^t$, which is a contradiction. Laplace transforms will work:

$$\begin{aligned} \mathscr{L}[x'(t)] - \mathscr{L}[x(t)] &= \mathscr{L}[e^t] \\ sX(s) - x(0) - X(s) &= 1/(s-1) \end{aligned}$$

$$\begin{aligned}(s-1)X(s) &= x(0) + 1/(s-1)\\ X(s) &= \frac{x(0)}{s-1} + \frac{1}{(s-1)^2}\\ x(t) &= x(0)e^t + \mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right].\end{aligned}$$

But we still need to calculate $\mathscr{L}^{-1}[1/(s-1)^2]$.

- There is another rule for this: If $F(s) = \mathscr{L}[f(t)]$ then

$$\mathscr{L}[t \cdot f(t)] = -F'(s).$$

Skip the proof for now. Applying this rule to $f(t) = e^{at}$ gives

$$\mathscr{L}[t \cdot e^{at}] = -\frac{d}{ds}\mathscr{L}[e^{at}] = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}.$$

Hence the equation $x'(t) - x(t) = e^t$ has solution

$$x(t) = x(0)e^t + \mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = x(0)e^t + t \cdot e^t.$$

This rule appears whenever there is a repeated factor in the denominator.

Mon, Mar 27

- Differentiation and integration are *linear operators*. That is, for any functions y_1, y_2 and constants c_1, c_2 we have

$$(c_1y_1 + c_2y_2)' = c_1y_1' + c_2y_2' \quad \text{and} \quad \int(c_1y_1 + c_2y_2) = c_1\int y_1 + c_2\int y_2.$$

- Remark: Problem 1 on Homework 4 is potentially confusing. Would I deliberately confuse you? Absolutely, yes. If you pass a course but you never felt confused then you probably didn't learn anything. If you get stuck on a problem and then later get unstuck then you are much more likely to remember this problem.

- The *Laplace transform* is a very special linear operator:

$$\begin{aligned}\mathscr{L}[f(t)] &= \int_0^\infty e^{-st}f(t)\,dt\\ \mathscr{L}[c_1f_1(t) + c_2f_2(t)] &= \int_0^\infty e^{-st}(c_1f_1(t) + c_2f_2(t))\,dt\\ &= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt\\ &= c_1\mathscr{L}[f_1(t)] + c_2\mathscr{L}[f_2(t)].\end{aligned}$$

It is a close relative of the *Fourier transform*:

$$\mathscr{F}[f(t)] = \int_{-\infty}^{\infty} e^{-i2\pi st} f(t)\, dt,$$

which is very common in applied mathematics. When we learn how to work with Laplace transforms we are also learning how to work with Fourier transforms.

- Recall the basic rules:

$$\begin{aligned}\mathscr{L}[e^{at}] &= 1/(s-a),\\ \mathscr{L}[f'(t)] &= s\mathscr{L}[f(t)] - f(0),\\ \mathscr{L}[f''(t)] &= s^2\mathscr{L}[f(t)] - sf(0) - f'(0).\end{aligned}$$

- Here is another rule:

$$\mathscr{L}[tf(t)] = -\frac{d}{ds}\mathscr{L}[f(t)].$$

Proof:

$$\begin{aligned}-\frac{d}{ds}\mathscr{L}[f(t)] &= -\frac{d}{ds}\int_0^{\infty} e^{-st} f(t)\, dt\\ &= -\int_0^{\infty} \frac{d}{ds} e^{-st} f(t)\, dt\\ &= -\int_0^{\infty} -te^{-st} f(t)\, dt\\ &= \int_0^{\infty} e^{-st} tf(t)\, dt\\ &= \mathscr{L}[tf(t)].\end{aligned}$$

From this we get

$$\begin{aligned}\mathscr{L}[t] &= \mathscr{L}[t\cdot 1] = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}\\ \mathscr{L}[t^2] &= \mathscr{L}[t\cdot t] = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3}\\ &\vdots\\ \mathscr{L}[t^n] &= \frac{n!}{s^{n+1}}.\end{aligned}$$

- Example: Solve $dy/dx = x + y$. Let $Y(s) = \mathscr{L}[y(x)]$ and apply $\mathscr{L}$ to both sides:

$$\begin{aligned}\mathscr{L}[dy/dx] &= \mathscr{L}[x] + \mathscr{L}[y]\\ sY(s) - y(0) &= \frac{1}{s^2} + Y(s)\\ sY(s) - Y(s) &= \frac{1}{s^2} + y(0)\end{aligned}$$

$$\begin{aligned}(s-1)Y(s) &= \frac{1}{s^2} + y(0)\\ Y(s) &= \frac{1}{s^2(s-1)} + \frac{y(0)}{s-1}.\end{aligned}$$

Then apply $\mathscr{L}^{-1}$ to both sides:

$$\begin{aligned}\mathscr{L}^{-1}[Y(s)] &= \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] + y(0)\mathscr{L}^{-1}\left[\frac{1}{s-1}\right]\\ y(x) &= \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] + y(0)e^x.\end{aligned}$$

Now we have a partial fractions problem. Since there is a repeated factor the Heaviside cover up method doesn't work, so we have to do it the long way:

$$\begin{aligned}\frac{1}{s^2(s-1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}\\ \frac{1}{s^2(s-1)} &= \frac{As(s-1)+B(s-1)+Cs^2}{s^2(s-1)}\\ 1 &= As(s-1)+B(s-1)+Cs^2\\ 0s^2+0s+1 &= (A+C)s^2+(B-A)s+(-B).\end{aligned}$$

Comparing coefficients gives $-B=1$, hence $B=-1$. Then $B-A=0$, hence $A=B=-1$. Then $A+C=0$, hence $C=-A=-(-1)=1$. We conclude that

$$\begin{aligned}\frac{1}{s^2(s-1)} &= \frac{-1}{s} + \frac{-1}{s^2} + \frac{1}{s-1}\\ \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] &= -\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \mathscr{L}^{-1}\left[\frac{1}{s^2}\right] + \mathscr{L}^{-1}\left[\frac{1}{s-1}\right]\\ &= -1-x+e^x.\end{aligned}$$

In summary, the equation $dy/dx = x+y$ has solution

$$y(x) = -1-x+e^x+y(0)e^x,$$

as we have seen several times before. Previously we used the method of integrating factors. This time it came down to a partial fractions computation.

Wed, Mar 29

- We discussed the Homework 5 solutions. Please read the solutions:

`https://www.math.miami.edu/~armstrong/311sp23/311sp23hw4sol.pdf`

Fri, Mar 31

- Reminder: The Laplace transform of a function $f(t)$ is defined as

  $$F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt.$$

  The operator $\mathscr{L}$ is linear because integration is linear.

- We should memorize the following formulas:
  - $\mathscr{L}[0] = 0$
  - $\mathscr{L}[1] = 1/s$
  - $\mathscr{L}[e^{at}] = 1/(s-a)$
  - $\mathscr{L}[f'(t)] = sF(s) - f(0)$
  - $\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$.

- Last time we saw a new general formula:

  $$\mathscr{L}[t \cdot f(t)] = -F'(s).$$

  This can be used to compute
  - $\mathscr{L}[t] = \mathscr{L}[t \cdot 1] = -(d/ds)\mathscr{L}[1] = -(d/ds)(1/s) = 1/s^2$
  - $\mathscr{L}[t^2] = \mathscr{L}[t \cdot t] = -(d/ds)\mathscr{L}[t] = -(d/ds)(1/s^2) = 2/s^3$
  - $\mathscr{L}[t^3] = \mathscr{L}[t \cdot t^2] = -(d/ds)\mathscr{L}[t^2] = -(d/ds)(2/s^3) = 6/s^4$, etc.

  We can also use it to compute

  $$\mathscr{L}[te^{at}] - \frac{d}{ds}\mathscr{L}[e^{at}] = -\frac{d}{ds} \cdot \frac{1}{s-a} = \frac{1}{(s-a)^2}, \quad \text{etc.}$$

- We have now seen the Laplace transform of most of the "elementary functions". The only functions missing are the trig functions:

  $$\mathscr{L}[\cos t] = ? \qquad \mathscr{L}[\sin t] = ?$$

  It is tedious to evaluate the integral:

  $$\mathscr{L}[\cos t] = \int_0^\infty e^{-st} \cos t\,dt = \text{ blah.}$$

  But we don't have to. Because of Euler's formula, trig functions are really just exponential functions in disguise. Recall that

  $$\begin{aligned} e^{it} &= \cos t + i \sin t, \\ e^{-it} &= \cos t - i \sin t, \end{aligned}$$

$$\begin{aligned}\cos t &= (e^{it} + e^{-it})/2,\\ \sin t &= (e^{it} - e^{-it}/2i.\end{aligned}$$

Hence we have

$$\begin{aligned}\mathscr{L}[\cos t] &= \frac{1}{2}\left(\mathscr{L}[e^{it}] + \mathscr{L}[e^{-it}]\right)\\ &= \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s-i}\right)\\ &= \frac{1}{2}\frac{(s+i)+(s-i)}{(s-i)(s+i)}\\ &= \frac{1}{2}\frac{2s}{s^2+1}\\ &= \frac{s}{s^2+1}\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}[\sin t] &= \frac{1}{2i}\left(\mathscr{L}[e^{it}] - \mathscr{L}[e^{-it}]\right)\\ &= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s-i}\right)\\ &= \frac{1}{2i}\frac{(s+i)-(s-i)}{(s-i)(s+i)}\\ &= \frac{1}{2i}\frac{2i}{s^2+1}\\ &= \frac{1}{s^2+1}.\end{aligned}$$

- Let's test these formulas on an easy problem. We know that the equation $x''(t)+x(t) = 0$ has solution $x(t) = x(0)\cos t + x'(0)\sin t$. Let's apply the Laplace transform:

$$\begin{aligned}x''(t) + x(t) &= 0\\ s^2X - sx(0) - x'(0) + X &= 0\\ (s^2+1)X &= sx(0) + x'(0)\\ X &= x(0)\frac{s}{s^2+1} + x'(0)\frac{1}{s^2+1}.\end{aligned}$$

Now apply the inverse transform:

$$\begin{aligned}x(t) &= \mathscr{L}^{-1}\left[x(0)\frac{s}{s^2+1} + x'(0)\frac{1}{s^2+1}\right]\\ &= x(0)\mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right] + x'(0)\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]\\ &= x(0)\cos t + x'(0)\sin t.\end{aligned}$$

Good, our formulas gave the correct answer.

- The same method of proof shows that

$$\mathscr{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \quad \text{and} \quad \mathscr{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}.$$

- Math should be useful or beautiful. There is no room in the world for ugly and useless mathematics. Since the Laplace transform is not beautiful, it must be useful. That is, it must help us solve a problem that we couldn't solve before. I see two main applications:
  - The Laplace transform naturally handles resonance.
  - The Laplace transform allows discontinuous inputs. (For example, hitting an object with a hammer or turning on a light switch.)

- **Resonance.** Consider the equation $x'' + x = \cos(\omega t)$ with $x(0) = x'(0) = 0$. Then

$$\begin{aligned} s^2 X + X &= \frac{s}{s^2+\omega^2} \\ X &= \frac{s}{(s^2+1)(s^2+\omega^2)} \\ x(t) &= \mathscr{L}^{-1}\left[\frac{s}{(s^2+1)(s^2+\omega^2)}\right]. \end{aligned}$$

As with every Laplace transform problem, this comes down to partial fractions. Our textbook rudely says (page 294) that "we can find without difficulty that"

$$\frac{s}{(s^2+1)(s^2+\omega^2)} = \frac{1}{\omega^2-1}\left[\frac{s}{s^2+1} - \frac{s}{s^2+\omega^2}\right].$$

That depends on your point of view. If you're not fluent with partial fractions it actually is a bit difficult. With quadratic factors in the denominator, the partial fraction expansion has the form

$$\frac{s}{(s^2+1)(s^2+\omega^2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+\omega^2}.$$

Get a common denominator, expand, then solve for A, B, C, D. Skip the details for now. Hence we obtain

$$x(t) = \frac{1}{\omega^2-1}\left(\frac{s}{s^2+1} - \frac{s}{s^2+\omega^2}\right) = \frac{1}{\omega^2-1}\left(\cos t - \cos(\omega t)\right).$$

We can also write this as

$$x(t) = \frac{2}{\omega^2-1} \cdot \sin\left(\frac{\omega+1}{2} \cdot t\right) \sin\left(\frac{\omega-1}{2} \cdot t\right).$$

Note that the solution blows up when $\omega \approx 1$.

- What happens when $\omega = 1$? Then

$$x(t) = \mathscr{L}^{-1}\left[\frac{s}{(s^2+1)(s^2+1)}\right] = \mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right].$$

The expression $s/(s^2+1)^2$ cannot be simplified with partial fractions. But we recognize it as a derivative:

$$\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{2s}{(s^2+1)^2}.$$

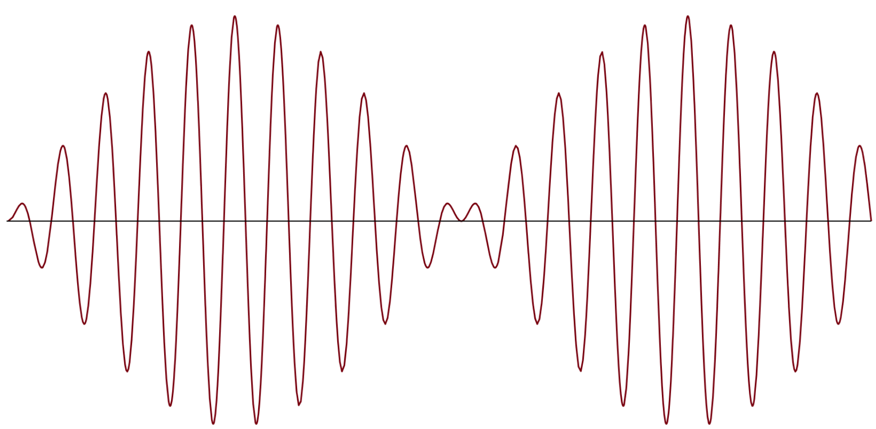
Since $\mathscr{L}[\sin t] = 1/(s^2+1)$ we see that

$$\mathscr{L}[t \cdot \sin t] = -\frac{d}{ds}\mathscr{L}[\sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = 2 \cdot \frac{s}{(s^2+1)^2}.$$

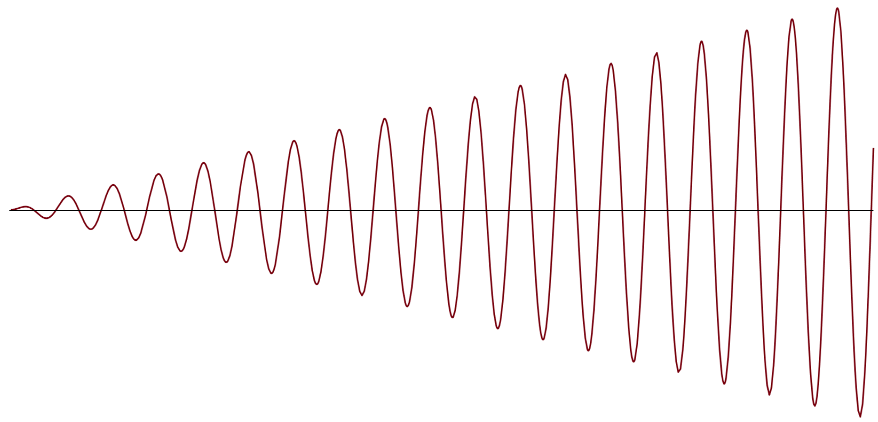
We conclude that

$$x(t) = \mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{t}{2} \cdot \sin t.$$

- Your washing machine has a natural vibrational frequency of 1. The drum rotates with variable frequency ω. As ω approaches 1 the washing machine starts to jump across the floor with amplitude $2/(\omega^2 - 1)$:



If $\omega = 1$ then the amplitude $t/2$ increases linearly until something bad happens:



April

Mon, Apr 3

- Method of Laplace Transforms:

$$\text{ODE} \quad \xrightarrow{\mathscr{L}} \quad \text{Algebra} \quad \xrightarrow{\text{partial fractions}} \quad \text{Algebra} \quad \xrightarrow{\mathscr{L}^{-1}} \quad \text{ODE}$$

- **Partial Fractions.** Let $p(s)$ be a polynomial of degree 1 let $q(s)$ be a polynomial of degree 2, where $p(s)$ and $q(s)$ do not share any common factors. Then

$$\frac{1}{p(s)^3 q(s)^2} = \frac{A}{p(s)} + \frac{B}{p(s)^2} + \frac{C}{p(s)^3} + \frac{Ds+E}{q(s)} + \frac{Fs+G}{q(s)^2}$$

for some constants A, B, C, D, E, F, G. For example, let $p(s) = s - 7$ and $q(s) = s^2 + 1$. Then

$$\frac{1}{(s-7)^3(s^2+1)^2} = \frac{A}{s-7} + \frac{B}{(s-7)^2} + \frac{C}{(s-7)^3} + \frac{Ds+E}{s^2+1} + \frac{Fs+G}{(s^2+1)^2}.$$

Get a common denominator, expand, and compare coefficients to get 7 equations in the 7 unknowns A, B, C, D, E, F, G. This is far too horrible to do by hand. **My computer says**

$$\frac{61}{781250(s-7)} - \frac{7}{31250\,(s-7)^2} + \frac{1}{2500\,(s-7)^3} + \frac{-61\,s-252}{781250(s^2+1)} + \frac{-73\,s-161}{62500\,(s^2+1)^2}$$

- Here's a medium example we can do by hand:

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$
$$\frac{1}{s^2(s^2+1)} = \frac{As(s^2+1) + B(s^2+1) + (Cs+D)s^2}{s^2(s^2+1)}$$

$$\begin{aligned}1 &= As(s^2+1) + B(s^2+1) + (Cs+D)s^2 \\ 0s^3 + 0s^2 + 0s + 1 &= (A+C)s^3 + (B+D)s^2 + (A+B)s + B.\end{aligned}$$

Comparing coefficients gives $A + C = 0$, $B + D = 0$, $A + B = 0$ and $B = 1$, so that $A = 0$, $B = 1$, $C = 0$ and $D = -1$, hence

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

- Alternative Method: Since the equation

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

is true for al values of s, we should substitute convenient values of s. Setting $s = 0$ gives $1 = 0 + B + 0$, hence $B = 1$. Setting $s = i$ makes $s^2 + 1$ so that $1 = 0 + 0 + (Ci + D)(-1)$. Similarly, setting $s = -i$ gives $1 = 0 + 0 + (-Ci + D)(-1)$. Then combining these two equations gives $C = 0$ and $D = -1$. At this point we have

$$1 = As(s^2+1) + (s^2+1) - s^2.$$

Substitute any value of s other than $s = 0$ or $s = \pm i$ to solve for A. This method is basically the Heaviside coverup method.

- By the way:

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] \\ &= \mathscr{L}^{-1}\left[\frac{1}{s^2}\right] - \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] = t - \sin t.\end{aligned}$$

- Recall the general formulas:

$$\mathscr{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \mathscr{L}[\cos(bt)] = \frac{s}{s^2+b^2} \qquad \mathscr{L}[\sin(bt)] = \frac{b}{s^2+b^2}.$$

- Old Problem: Consider the equation $x''(t) - 2x'(t) + 5x(t) = 0$ with characteristic polynomial $\lambda^2 - 2\lambda + 5$. The roots are

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i.$$

So the general solution is

$$x(t) = c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t} = e^t\left(c_1 e^{i2t} + c_2 e^{-i2t}\right) = e^t\left(c_3\cos(2t) + c_4\sin(2t)\right).$$

With initial conditions $x(0) = 0$ and $x'(0) = 1$ we get $x(t) = e^t \sin(2t)/2$.

- Let's check that Laplace Transforms give the same answer:

$$\begin{aligned} \mathscr{L}[x''(t)] - 2\mathscr{L}[x'(t)] + 5\mathscr{L}[x(t)] &= 0 \\ s^2X - sx(0) - x'(0) - 2(sX - x(0)) + 5X &= 0 \\ (s^2 - 2s + 5)X &= sx(0) + x'(0). \end{aligned}$$

For simplicity let $x(0) = 0$ and $x'(0) = 1$ as in the previous example. Then

$$\begin{aligned} (s^2 - 2s + 5)X &= 1 \\ X &= \frac{1}{s^2 - 2s + 5}. \end{aligned}$$

So I guess we must have

$$\mathscr{L}^{-1}\left[\frac{1}{s^2 - 2s + 5}\right] = \frac{1}{2} \cdot e^t \cdot \sin(2t).$$

But why?

- Method 1:

$$\frac{1}{s^2 - 2s + 5} = \frac{1}{(s - (1+2i))(s - (1-2i))} = \cdots = \frac{-i/4}{s - (1+2i)} + \frac{i/4}{s - (1-2i)}.$$

The inverse transform is

$$-\frac{i}{4} \cdot \mathscr{L}^{-1}\left[\frac{1}{s - (1+2i)}\right] + \frac{i}{4} \cdot \mathscr{L}^{-1}\left[\frac{1}{s - (1-2i)}\right] = -\frac{i}{4}e^{(1+2i)t} + \frac{i}{4}e^{(1-2i)t},$$

which simplifies via Euler's formula.

- Method 2 avoids complex numbers: Complete the square in $s^2 - 2s + 5$ to get

$$s^2 - 2s + 5 = s^2 - 2s + 1 - 1 + 5 = (s^2 - 2s + 1) + 4 = (s-1)^2 + 4.$$

Then

$$\mathscr{L}^{-1}\left[\frac{1}{s^2 - 2s + 5}\right] = \mathscr{L}^{-1}\left[\frac{1}{(s-1)^2 + 2^2}\right] = \frac{1}{2} \cdot \mathscr{L}^{-1}\left[\frac{2}{(s-1)^2 + 2^2}\right] = \quad ?$$

This looks a lot like $\sin(2t)/2$, but instead of s^2 we have $(s-1)^2$. There is a general rule for this:

If $F(s) = \mathscr{L}[f(t)]$ then $\mathscr{L}[e^{at} \cdot f(t)] = F(s - a)$.

In our case, we have $\mathscr{L}[\sin(2t)] = 2/(s^2 + 2^2) = F(s)$, so

$$\mathscr{L}[e^t \cdot \sin(2t)] = F(s-1) = \frac{2}{(s-1)^2 + 2^2}.$$

Finally,

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 - 2s + 5}\right] = \frac{1}{2} \cdot \mathscr{L}^{-1}\left[\frac{2}{(s-1)^2 + 2^2}\right] = \frac{1}{2} \cdot e^t \cdot \sin(2t).$$

Wed Apr 5

- Let $F(s) = \mathscr{L}[f(t)]$. Recall:

$$\begin{aligned} \mathscr{L}[f'(t)] &= sF(s) - f(0), \\ \mathscr{L}[t \cdot f(t)] &= -F'(s), \\ \mathscr{L}[e^{at} \cdot f(t) &= F(s-a). \end{aligned}$$

  Also recall:

$$\mathscr{L}[\cos(kt)] = \frac{s}{s^2+k^2} \qquad \text{and} \qquad \mathscr{L}[\sin(kt)] = \frac{k}{s^2+k^2}.$$

  Combining these formulas gives, e.g.,

$$\mathscr{L}[e^{at} \cdot \sin(kt)] = \mathscr{L}[\sin(kt)]_{s \to s-a} = \left.\frac{k}{s^2+k^2}\right|_{s \to s-a} = \frac{k}{(s-a)^2+k^2}$$

  and

$$\begin{aligned} \mathscr{L}[t \sin(kt)] &= -\frac{d}{s}\left(\frac{k}{s^2+k^2}\right) \\ &- k\frac{d}{ds}(s^2+k^2)^{-1} \\ &- k(-1)(s^2+k^2)^{-2}(2s) \\ &= \frac{2ks}{(s^2+k^2)^2} \end{aligned}$$

- This last formula shows up in resonance problems. For example:

$$\begin{aligned} x''(t) + x(t) &= \cos t \\ s^2 X - sx(0) - x'(0) + X &= \frac{s}{s^2+1} \\ (s^2+1)X &= sx(0) + x'(0) + \frac{s}{s^2+1} \\ X &= x(0)\frac{s}{s^2+1} + x'(0)\frac{1}{s^2+1} + \frac{s}{(s^2+1)^2} \\ x(t) &= x(0)\cos t + x'(0)\sin t + \mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]. \end{aligned}$$

  But we just saw that

$$\mathscr{L}[t \cdot \sin t] = \frac{2s}{(s^2+1)}, \qquad \text{hence } \mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2} \cdot t \cdot \sin t.$$

- The very similar equation $x''(t) + x(t) = \sin t$ has solution

$$x(t) = x(0)\cos t + x'(0)\sin t + \mathscr{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right].$$

But we have not yet seen a function $f(t)$ with $\mathscr{L}[f(t)] = 1/(s^2+1)^2$. This is much more annoying than it looks, so I will just quote the formula from the table:

$$\mathscr{L}^{-1}\left[\frac{1}{(s^2+k^2)}\right] = \frac{1}{2k^3}\left(\sin(kt) - kt\cos(kt)\right).$$

- Discussion: In the 1200s Fibonacci introduced the decimal system to Europe. This was seen as a business technology that sped up financial computations. Businesses hired human "computers" to just add and multiply large numbers. Since multiplication is much more time consuming than addition, Napier invented logarithms in the 1600s to convert multiplication problems into addition. Take a and b. Look up the numbers $\log(a)$ and $\log(b)$ in a table. Add to get $\log(a) + \log(b)$. Then do a reverse look up to find ab. This works because

$$\log^{-1}\left(\log(a) + \log(b)\right) = ab.$$

  Napier made a profit by selling a table of logarithms. Heaviside's method of Laplace transforms follows the same idea. Use a table to convert a differential equation into an algebraic equation. Solve the algebra. Use the table to convert back to the differential equation.

- Usually there are multiple ways a differential equation. Laplace transform methods can be fast for the experienced user but they are not always the best method. However, there is one type of problem that is perfectly suited to Laplace transforms: **discontinuous inputs**, such as a hammer hitting an object or a light switch being turned on.

- The *Heaviside step function* is defined as follows:

$$H(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 0, \end{cases} \qquad \text{hence} \qquad H(t-a) = \begin{cases} 0 & t < a, \\ 1 & t > a. \end{cases}$$

  Think of $H(t-a)$ as a light switch that turns on at time $t = a$. Let $F(s) = \mathscr{L}[f(t)]$ for some function $f(t)$ and let $a \geq 0$. Then we have

$$\mathscr{L}[H(t-a)\cdot f(t-a)] = e^{-as}F(s).$$

  Proof: Note that

$$H(t-a)\cdot f(t-a) = \begin{cases} 0 & t < a, \\ f(t-a) & t > a. \end{cases}$$

  Hence

$$\begin{aligned} \mathscr{L}[H(t-a)\cdot f(t-a)] &= \int_0^\infty e^{-st}H(t-a)\cdot f(t-a)\,dt \\ &= \int_0^a 0\,dt + \int_a^t e^{-st}f(t-a)\,dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-a(\tau+a)} f(\tau)\, d\tau && \tau = t-a \\
&= e^{-as} \int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau \\
&= e^{-as} F(s).
\end{aligned}
$$

- The *Dirac delta function* is the "derivative" of the step function.[3] It has the following strange properties:

$$
\delta(t) = \begin{cases} 0 & t \neq 0, \\ \text{undefined} & t = 0, \end{cases} \quad \text{and} \quad \int_a^b \delta(t)\, dt = \begin{cases} 1 & a \leq 0 \leq b, \\ 0 & \text{otherwise.} \end{cases}
$$

The graph of $\delta(t)$ is an infinitely thin rectangle of area 1 sitting above $t = 0$. We use it to model an "impulse", or an instantaneous transfer of energy. For $a \geq 0$, we have

$$
\mathscr{L}[\delta(t-a)] = e^{-as}.
$$

- Quick example: A spring is hit by a hammer at time a:

$$
\begin{aligned}
x''(t) + x(t) &= \delta(t-a) \\
s^2 X - sx(0) - x'(0) + X &= e^{-as} \\
(s^2+1)X &= sx(0) + x'(0) + e^{-as} \\
X &= x(0)\frac{s}{s^2+1} + x'(0)\frac{1}{s^2+1} + e^{-as}\frac{1}{s^2+1} \\
x(t) &= x(0)\cos t + x'(0)\sin t + H(t-a)\sin(t-a).
\end{aligned}
$$

You will finish analyzing this problem on the homework.

Fri, Apr 7

- The *Heaviside step function*

$$
H(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 1 \end{cases}
$$

allow us to model piecewise-defined and discontinuous functions. For example, consider the function

$$
f(t) = \begin{cases} -t & t < 2, \\ t^2 - 5 & t > 2. \end{cases}
$$

Then we have

$$
f(t) = -t + H(t-2)(t^2 - 5 + t).
$$

[3]Dirac introduced $\delta(t)$ in his study of quantum mechanics. Mathematics are always careful to note that $\delta(t)$ isn't **really** a function, because any actually function that is zero almost everywhere should have integral zero. However, $\delta(t)$ is a very convenient notation and it gives perfectly good results. So don't worry.

- The derivative of the step function is a slightly fictional function, called the *Dirac delta function*:
$$H'(t) = \delta(t) \quad \text{and} \quad H'(t-a) = \delta(t-a).$$
As a function it is just
$$\delta(t-a) = \begin{cases} 0 & t \neq a, \\ +\infty & t = a. \end{cases}$$
But it has the strange property that $\int \delta(t-a)\, dt = 1$. Intuition: The graph of $\delta(t-a)$ is an infinitely tall and infinitely skinny rectangle of area 1 sitting above $t = a$.

- The delta function is also called the *unit impulse*. We can use it to model instantaneous transfer of energy in a physical system, such as a hammer blow or the closing of an electrical switch.

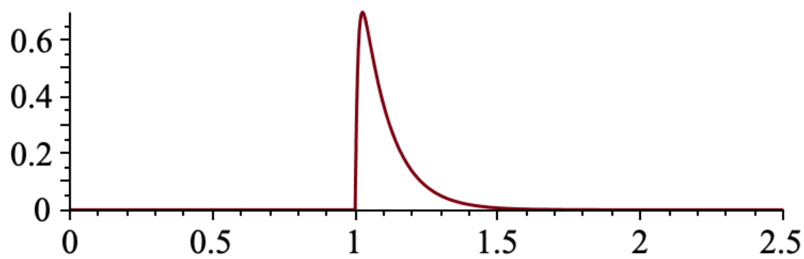
- The step function $H(t)$ is named after Heaviside because he realized that it plays well with Laplace transforms. Last time we proved:
$$\text{If } F(s) = \mathscr{L}[f(t)] \text{ and } a \geq 0 \text{ then } \mathscr{L}[e^{-as}F(s)] = H(t-a)f(t-a).$$
In particular, since $\mathscr{L}[1] = 1/s$ we have
$$\mathscr{L}[H(t-a)] = \mathscr{L}[H(t-a)\cdot 1] = e^{-as}\mathscr{L}[1] = e^{-as}/s.$$
And since $\delta(t-a) = H'(t-a)$ we have
$$\mathscr{L}[\delta(t-a)] = \mathscr{L}[H'(t-a)] = s\mathscr{L}[H(t-a)] - H(0-a) = s \cdot e^{-as}/s = e^{-as}.$$
Interesting special case:[4]
$$\boxed{\mathscr{L}[\delta(t)] = 1.}$$

- Application: A hockey puck sits on the ice at position $x = 0$. At time $t = 0$ a hockey player hits the puck with an instantaneous force of F_0. The equation of motion is
$$mx''(t) + \gamma x'(t) = F_0\delta(t),$$
where m is the mass of the puck and γ is the friction of the ice. Apply Laplace transforms:
$$\begin{aligned} m\mathscr{L}[x''(t)] + \gamma\mathscr{L}[x'(t)] &= F_0\mathscr{L}[\delta(1)] \\ m(s^2X - s0 - 0) + \gamma(sX - 0) &= F_0 \cdot 1 \\ (ms^2 + \gamma s)X &= F_0 \\ X &= \frac{F_0}{s(ms+\gamma)} \\ x(t) &= F_0 \cdot \mathscr{L}^{-1}\left[\frac{1}{s(ms+\gamma)}\right]. \end{aligned}$$
How far does the hockey puck travel before it is stopped by friction? Homework.

[4] Actually, there is something funny here because $H(0)$ is not defined. Don't worry about it. Just don't worry about anything when it comes to delta functions.

- Application: RLC circuits. Let R, L, C be the resistance, inductance and capacitance of an electric circuit. Let $e(t)$ be the applied voltage at time t. Then the current $i(t)$ satisfies
$$Li''(t) + Ri'(t) + \frac{1}{C}i(t) = e'(t).$$
This is analogous to the spring equation where L is the inertia, R is the friction and $1/C$ is the stiffness of the spring. Suppose that $e(t)$ comes from a V volt battery that is switched on at time a, so that
$$e(t) = \begin{cases} 0 & t < a, \\ V & t > a, \end{cases} \qquad \text{and} \qquad e'(t) = V\delta(t-a).$$

- Textbook problem, page 309: Let $L = 110$, $L = 1$, $C = 0.001$, $V = 90$, $a = 1$, so
$$i''(t) + 110i'(t) + 1000i(t) = 90\delta(t-1).$$
Suppose the circuit starts in equilibrium, so $i(0) = i'(0) = 0$. Then
$$\begin{aligned} s^2I + 110sI + 1000I &= 90e^{-s} \\ (s^2 + 110s + 1000)I &= 90e^{-s} \\ I &= e^{-s}\frac{90}{s^2 + 100s + 1000} \\ I &= e^{-s}\frac{90}{(s+10)(s+100)} \\ I &= e^{-s}\left(\frac{1}{s+10} - \frac{1}{s+100}\right). \end{aligned}$$
This is how you know it's a textbook problem. Note that
$$\mathscr{L}^{-1}\left[\frac{1}{s+10} - \frac{1}{s+100}\right] = e^{-10t} - e^{-100t}.$$
Now we apply the rule $\mathscr{L}[e^{-as}f(t)] = H(t-a)F(t-a)$ to get
$$\begin{aligned} i(t) &= H(t-1)\left(e^{-10(t-1)} - e^{-100(t-1)}\right) \\ &= \begin{cases} 0 & t < 0, \\ e^{-10(t-1)} - e^{-100(t-1)} & t > 1. \end{cases} \end{aligned}$$
I certainly can't see the graph of $i(t)$ in my head. We could use Calc I curve sketching techniques, but I just used my computer:



The current spikes when the switch is closed but returns quickly to zero. Interpretation: The battery causes current that charges the capacitor. When the capacitor is fully charged the current stops. In our case we had $R^2 - 4L/C > 0$. If $R^2 - 4L/C < 0$ then the polynomial $Ls^2 + Rs + 1/C$ has imaginary roots and the current will oscillate as it goes to zero.

Mon, Apr 10

- Recall: $\mathscr{L}[\delta(t)] = 1$. Hockey puck on ice:

$$\begin{aligned}
\text{(force)} &= \text{(friction)} + \text{(unit impulse at } t = 0) \\
mx''(t) &= -\gamma x'(t) + \delta(t) \\
ms^2 X &= -\gamma s X + 1 && \text{assume } x(0) = x'(0) = 0 \\
s(ms+\gamma)X &= 1 \\
X &= \frac{1}{s(ms+\gamma)}.
\end{aligned}$$

  Homework: Find the inverse transform. How far does the puck go before stopping?

- Continued from last time: problem from page 309. An RLC circuit has equation

$$i''(t) + 110i'(t) + 1000i(t) = e'(t),$$

  where the applied voltage is

$$e(t) = \begin{cases} 90 & 0 < t < 1, \\ 0 & t > 1. \end{cases}$$

  (A 90 volt battery is switched on at $t = 0$ and switched off at $t = 1$.) In terms of the Heaviside function:

$$e(t) = 90(H(t) - H(t-1)).$$

  The derivative is

$$e'(t) = 90(H'(t) - H'(t-1)) = 90(\delta(t) - \delta(t-1)).$$

The term $90\delta(t)$ says that $e(t)$ jumps from 0 to 90 at $t = 0$. The term $-90\delta(t-1)$ says that $e(t)$ jumps from 90 to 0 at $t = 1$. Assume that $i(0) = i'(0) = 0$ then apply transforms:

$$\begin{aligned} i''(t) + 110i'(t) + 1000i(t) &= 90(\delta(t) - \delta(t-1)) \\ s^2 I + 110sI + 1000I &= 90(1 - e^{-s}). \end{aligned}$$

(Recall: $\mathscr{L}[\delta(t-a)] = e^{-as}$.) Solve for I:

$$\begin{aligned} s^2 I + 110sI + 1000I &= 90(1 - e^{-s}) \\ (s^2 + 110s + 1000)I &= 90(1 - e^{-s}) \\ I &= \frac{90}{(s^2 + 110s + 1000)}(1 - e^{-s}) \\ &= \frac{90}{(s+10)(s+100)}(1 - e^{-s}). \end{aligned}$$

(The denominator has a nice factorization because this is a textbook problem.) Now compute the partial fractions:

$$\begin{aligned} \frac{90}{(s+10)(s+100)} &= \frac{A}{s+10} + \frac{B}{s+100} \\ \frac{90}{(s+10)(s+100)} &= \frac{A(s+100) + B(s+10)}{(s+10)(s+100)} \\ 90 &= A(s+100) + B(s+10). \end{aligned}$$

Substitute $s = -10$ to get $90 = 90A$ and substitute $s = -100$ to get $90 = -90B$, hence

$$\frac{90}{(s+10)(s+100)} = \frac{1}{s+10} - \frac{1}{s+100}.$$

Continuing from above:

$$\begin{aligned} I(s) &= \left(\frac{1}{s+10} - \frac{1}{s+100}\right)(1 - e^{-s}) \\ &= \frac{1}{s+10} - \frac{1}{s+100} - \frac{e^{-s}}{s+10} + \frac{e^{-s}}{s+100} \\ i(t) &= e^{-10t} - e^{-100t} - \mathscr{L}^{-1}\left[\frac{e^{-s}}{s+10}\right] + \mathscr{L}^{-1}\left[\frac{e^{-s}}{s+100}\right]. \end{aligned}$$

Recall the general rule: If $F(s) = \mathscr{L}[f(t)]$ then

$$\mathscr{L}^{-1}\left[e^{-as}F(s)\right] = H(t-a)f(t-a) = \begin{cases} 0 & t < a, \\ f(t-a) & t > a, \end{cases}$$

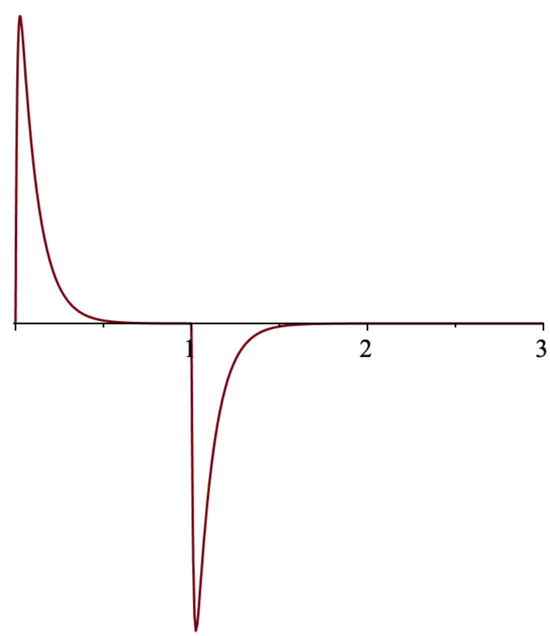
where $H(t)$ is the Heaviside step function:

$$H(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 0. \end{cases}$$

We conclude that

$$
\begin{aligned}
i(t) &= e^{-10t} - e^{-100t} - \mathscr{L}^{-1}\left[\frac{e^{-s}}{s+10}\right] + \mathscr{L}^{-1}\left[\frac{e^{-s}}{s+100}\right] \\
&= e^{-10t} - e^{-100t} - H(t-1)e^{-10(t-1)} + H(t-1)e^{-100(t-1)} \\
&= \begin{cases} 0 & t < 0, \\ e^{-10t} - e^{-100t} & 0 < t < 1, \\ e^{-10t} - e^{-100t} - e^{-10(t-1)} + e^{-100(t-1)} & t > 1. \end{cases}
\end{aligned}
$$

Here is a graph:



When the switch is closed at $t = 0$ the current spikes (clockwise) while the capacitor charges. When the switch is open at $t = 1$ the current spikes (anticlockwise) while the capacitor discharges.

- **Coupled Oscillators.** See the picture on page 281. Two masses m_1 and m_2 are attached by springs. At time t the first spring is stretched by $x(t)$ and the second is stretched by $y(t) - x(t)$. The first mass fees two spring forces and the second mass feels one spring force:

$$
\begin{cases} m_1 x''(t) &= -k_1 x(t) + k_2(y(t) - x(t)), \\ m_2 y''(t) &= -k_2(y(t) - x(t)). \end{cases}
$$

This is our first example of a **system of differential equations.** We will solve this system using Laplace transforms. For simplicity, the textbook uses $m_1 = 2$, $m_2 = 1$,

$k_1 = 4$ and $k_2 = 2$ and assumes that the springs start at rest: $x(0) = x'(0) = y(0) = y'(0) = 0$. Let's also hit the second mass with a unit impulse (to the left) at time $t = 0$:

$$\begin{cases} 2x'' &= -4x + 2(y - x), \\ y'' &= -2(y - x) - \delta(t), \end{cases}$$
$$\begin{cases} x'' &= -3x + y, \\ y'' &= 2x - 2y - \delta(t). \end{cases}$$

Apply Laplace transforms to get

$$\begin{cases} s^2X &= -3X + Y, \\ s^2Y &= 2X - 2Y - 1. \end{cases}$$

Now we can use algebra to solve for X and Y. Finally, we will compute the inverse transforms of X and Y to get $x(t)$ and $y(t)$. Details next time.

Wed, Apr 12

- Continued from last time: We have a system of two second order linear ODEs with constant coefficients:

$$\begin{cases} x'' &= -3x + y, \\ y'' &= 2x - 2y - \delta(t). \end{cases}$$

  These equations are coupled because the equation for x'' involves y and the equation for y'' involves x. We derived these equations from a pair of masses connected by springs.

- For the rest of this course my goal is to examine similar "systems" of differential equations. Linear algebra provides a powerful method for dealing with such systems. Since linear algebra is not a pre-requisite for this course I will keep the discussion very explicit and example-based.

- Remark: I was educated in Canada, where every student of science and engineering is required to take two semesters of linear algebra in the first year, in parallel with two semesters of calculus. I don't know why the Canadian and American systems are so different. Do you have any ideas?

- Before diving into linear algebra, let me show you that the method of Laplace transforms is strong enough to solve the above system. The computations are a bit tedious, but it can be done. We assume that the masses start at rest:

$$x(0) = x'(0) = y(0) = y'(0) = 0.$$

  Apply Laplace transforms to get

$$\begin{cases} s^2X &= -3X + Y, \\ s^2Y &= 2X - 2Y - 1, \end{cases} \quad \rightsquigarrow \quad \begin{cases} (s^2+3)X - Y &= 0, \\ -2X + (s^2+2)Y &= -1. \end{cases}$$

(Recall that $\mathscr{L}[\delta(t)] = 1$.) To eliminate Y we can multiply the first equation by s^2+2 then add the equations:

$$\begin{array}{rl} (s^2+2)(s^2+3)X - \cancel{(s^2+2)Y} & = 0, \\ -2X + \cancel{(s^2+2)Y} & = -1, \\ \hline [(s^2+2)(s^2+3)-2]X + 0 & = -1. \end{array}$$

This becomes

$$\begin{aligned} [(s^2+2)(s^2+3)-2]X &= -1 \\ (s^4+5s^2+6-2)X &= -1 \\ (s^4+5s^2+4)X &= -1 \\ (s^2+1)(s^2+4)X &= -1 \\ X &= \frac{-1}{(s^2+1)(s^2+4)}. \end{aligned}$$

The denominator factored because this is a carefully chosen textbook problem. Now we apply partial fractions to get

$$\begin{aligned} X &= -\frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{1}{6}\cdot\frac{2}{s^2+4} \\ x(t) &= \mathscr{L}^{-1}\left[-\frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{1}{6}\cdot\frac{2}{s^2+4}\right] \\ &= -\frac{1}{3}\cdot\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] + \frac{1}{6}\cdot\mathscr{L}^{-1}\left[\frac{2}{s^2+4}\right] \\ &= -\frac{1}{3}\cdot\sin(t) + \frac{1}{6}\cdot\sin(2t). \end{aligned}$$

A similar method gives

$$Y = \frac{s^2+3}{(s^2+1)(s^2+4)} = -\frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{1}{6}\cdot\frac{2}{s^2+4}$$

and hence

$$y(t) = -\frac{2}{3}\cdot\sin(t) - \frac{1}{6}\cdot\sin(2t).$$

Thus we have explicitly solved a coupled system of two differential equations. Here is an animated gif of the solution

`https://www.math.miami.edu/~armstrong/311sp23/coupled_oscillators.gif`

- For the rest of today I will give an **advertisement** for using linear algebra to solve differential equations. I will not explain the details right away. Consider the following system of coupled first order equations from page 369 of the text:

$$\begin{cases} x'(t) &= 4x(t) + 2y(t), \\ y'(t) &= 3x(t) - y(t). \end{cases} \qquad (*)$$

This can be solved with Laplace transforms but we're not going to do that. Instead, we will write these two equations as a single matrix equation:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

To make this look better, we will rewrite the vector (x, y) with the single single boldface letter $\mathbf{x}$, so that

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

We will also express the matrix of coefficients with a singe letter:

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$

Thus we can express the complicated system $(*)$ very succinctly:

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

- Why bother? There is a deep analogy here. Recall that the single equation

  $$x(t) = ax'(t) \quad \text{has solution} \quad x(t) = x(0)e^{at}.$$

  Amazingly, the vector equation

  $$\mathbf{x}(t) = A\mathbf{x}'(t) \quad \text{has solution} \quad \mathbf{x}(t) = e^{At}\mathbf{x}(0),$$

  where $\mathbf{x}(0) = (x(0), y(0))$ is the vector of initial conditions and $e^{At} = \exp(At)$ is the "exponential matrix". Exponential matrices are difficult to compute by hand, but my computer tells me that

  $$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \quad \rightsquigarrow \quad \exp(At) = \begin{pmatrix} \frac{1}{7} \cdot e^{-2t} + \frac{6}{7} \cdot e^{5t} & -\frac{1}{7} \cdot e^{-2t} + \frac{2}{7} \cdot e^{5t} \\ -\frac{1}{7} \cdot e^{-2t} + \frac{3}{7} \cdot e^{5t} & \frac{6}{7} \cdot e^{-2t} + \frac{1}{7} \cdot e^{5t} \end{pmatrix}.$$

  Hence the solution to our system of equations is

  $$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{7} \cdot e^{-2t} + \frac{6}{7} \cdot e^{5t} & -\frac{1}{7} \cdot e^{-2t} + \frac{2}{7} \cdot e^{5t} \\ -\frac{1}{7} \cdot e^{-2t} + \frac{3}{7} \cdot e^{5t} & \frac{6}{7} \cdot e^{-2t} + \frac{1}{7} \cdot e^{5t} \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

- That's a bit hard to read. If I want to solve the system by hand I will use the method of eigenvalues. The preliminary work consists of discovering the *eigenvectors* and *eigenvectors* of the matrix A. The result of these computations will be two vector equations:

  $$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$
  $$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

From these two equations I can read off the general solution:

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= c_1 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^{-2t} + 2c_2 e^{5t} \\ -3c_1 e^{-2t} + c_2 e^{5t} \end{pmatrix},\end{aligned}$$

for some constants c_1 and c_2, which can be determined from the initial conditions.

- As I said, this is just an advertisement. I will explain the details soon. My plan is to teach you how to compute eigenvectors and eigenvalues of small matrices, then read off the solution to a small system of ODEs. Sadly, there is not much time to explain the ideas behind the computations.

Fri, Apr 14

- We discussed the Homework 5 solutions. Please read the solutions:

  `https://www.math.miami.edu/~armstrong/311sp23/311sp23hw5sol.pdf`

Mon, Apr 17

- This week we will discuss what we can from Chapter 5. Next week we will review for Exam 2.

- Chapter 5 is about *systems of linear differential equations*. The most important prerequisite for this chapter is the definition of *matrix multiplication*. In particular, we need to know how to multiply a matrix and a vector to obtain a vector:

$$\begin{gathered}(\text{matrix})\cdot(\text{vector}) = (\text{vector}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.\end{gathered}$$

  It is common to denote matrices with uppercase letters and vectors with boldface lowercase letters. For example, if we write

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  then we have

$$A\mathbf{x} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

- Using this notation we can express a system of linear differential equations as a single matrix differential equation:

$$\begin{cases} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t), \end{cases}$$

$$\begin{aligned}
&\rightsquigarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
&\rightsquigarrow \mathbf{x}'(t) = A\mathbf{x}(t).
\end{aligned}$$

- Remark: If $\mathbf{x}(t)$ is a vector of functions, the vector of derivatives $\mathbf{x}'(t)$ is defined by taking the derivative of each entry.

- Every system of linear differential equations, no matter how general, can be expressed in the form
$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

- At this level of generality, the solution is easy to write down:
$$\mathbf{x}(t) = e^{At}\mathbf{x}(0),$$
where $\mathbf{x}(0)$ is the vector of initial conditions and e^{At} is the "matrix exponential". The matrix exponential is difficult to compute by hand. (A computer has no trouble.)

- Luckily we don't need to compute the matrix exponential. When working by hand, we will look for basic solutions of the form
$$\mathbf{x}(t) = e^{\lambda t}\mathbf{c},$$
where λ is a number called an *eigenvalue* and $\mathbf{c}$ is a constant vector called an *eigenvector*. Substituting this guess into the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ gives the *eigenvalue equation*:
$$\begin{aligned}
\mathbf{x}'(t) &= A\mathbf{x}(t) \\
(e^{\lambda t}\mathbf{c})' &= A(e^{\lambda}t)\mathbf{c} \\
\lambda e^{\lambda t}\mathbf{c} &= e^{\lambda t}A\mathbf{c} \\
\lambda\mathbf{c} &= A\mathbf{c}.
\end{aligned}$$
Remark: In this computation we used the "linearity" of vector differentiation and matrix multiplication. Basically, rules that look true remain true in this context.

- The main computation of Chapter 5 is to solve the eigenvector equation $A\mathbf{c} = \lambda\mathbf{c}$. I will illustrate the method using the example from last time. We are looking for numbers λ, x, y satisfying
$$\begin{aligned}
\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \lambda \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} 4x + 2y \\ 3x - y \end{pmatrix} &= \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \\
\begin{pmatrix} 4x + 2y - \lambda x \\ 3x - y - \lambda y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} (4 - \lambda)x + 2y \\ 3x + (-1 - \lambda)y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a random value of λ this system has only the boring solution $x = y = 0$. To get an interesting solution, the coefficient matrix must have "determinant zero":

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} &= 0 \\ (4-\lambda)(-1-\lambda) - (2)(3) &= 0 \\ -4 + \lambda - 4\lambda + \lambda^2 - 6 &= 0 \\ \lambda^2 - 3\lambda - 10 &= 0 \\ (\lambda + 2)(\lambda - 5) &= 0. \end{aligned}$$

We conclude that this matrix has two eigenvalues: $\lambda = -2$ and $\lambda = 5$. In general, an $n \times n$ matrix has n eigenvalues. (Repeated eigenvalues require special methods, as in Chapter 2.)

- Remark: The *determinant* of a 2×2 matrix is defined as follows:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

It is a theorem from linear algebra that the vector equation $A\mathbf{x} = \mathbf{0}$ has a nonzero solution $\mathbf{x}$ if and only if $\det(A) = 0$.

- Now that we have the eigenvalues $\lambda = -2$ and $\lambda = 5$ we can compute the corresponding eigenvectors.

- For $\lambda = -2$ we need to solve

$$\begin{aligned} \begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4-(-2) & 2 \\ 3 & -1-(-2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 6x + 2y \\ 3x + y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Note that the two equations $6x + 2y = 0$ and $3x + y = 0$ are actually the same! This is good news because it means that $x = y = 0$ is not the only solution. In fact, we get infinitely many solutions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ -6 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ or } \begin{pmatrix} 1/2 \\ -3/2 \end{pmatrix} \text{ or } \cdots$$

It doesn't matter which one we pick. Usually I try to minimize the size of the entries without introducing fractions. In this case I'll pick $(1, -3)$. Thus we have the desired eigenvector. Check:

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4(1) + 2(-3) \\ 3(1) - 1(-3) \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Taking $\lambda = -2$ and $\mathbf{c} = (1, -3)$ gives a basic solution to the differential equation $\mathbf{x}'(t) = A\mathbf{x}(t)$:

$$\begin{aligned} \mathbf{x}(t) &= e^{\lambda t}\mathbf{c} \\ \mathbf{x}(t) &= e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix}. \end{aligned}$$

- To find another basic solution we will find an eigenvector with eigenvalue $\lambda = 5$:

$$\begin{aligned} \begin{pmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4-(5) & 2 \\ 3 & -1-(5) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -x + 2y \\ 3x - 6y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

  Again, this system has infinitely many solutions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} -6 \\ -3 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \text{ or } \cdots$$

  Taking $\mathbf{c} = (2, 1)$ gives the basic solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{x}(t) = e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{5t} \\ e^{5t} \end{pmatrix}.$$

- Putting everything together: The system

$$\begin{cases} x'(t) &= 4x(t) + 2y(t), \\ y'(t) &= 3y(t) - 1y(t) \end{cases}$$

  has general solution

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= ae^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + be^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} ae^{-2t} + 2be^{5t} \\ -3ae^{-2t} + be^{5t} \end{pmatrix} \end{aligned}$$

  for any constants a and b.

- The specific values of a and b depend on the initial conditions $x(0)$ and $y(0)$. For example, let's take $x(0) = 7$ and $y(0) = 0$. Substituting $t = 0$ in the general solution gives a system of two equations for a and b:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} ae^0 + 2be^0 \\ -3ae^0 + be^0 \end{pmatrix}$$
$$\begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} a + 2b \\ -3a + b \end{pmatrix}.$$

After a bit of work we find $a = 1$ and $b = 3$. Finally, the solution of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ with initial conditions $x(0) = 7$ and $y(0) = 0$ is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 1e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 3e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} e^{-2t} + 6e^{5t} \\ -3e^{-2t} + 3e^{5t} \end{pmatrix},$$

i.e.,

$$x(t) = e^{-2t} + 6e^{5t} \quad \text{and} \quad y(t) = -3e^{-2t} + 3e^{5t}.$$

- This same basic method applies to any system of linear differential equations. There just two issues:
  - It is hard to find the eigenvalues when the system involves many equations. In practice, the eigenvalues of large matrices can only be approximated.
  - Repeated eigenvalues introduce complications. In general, if the eigenvalue λ appears twice then the corresponding basic solution has the form

    $$\mathbf{x}(t) = (\mathbf{c}_1 + t\mathbf{c}_2)e^{\lambda t},$$

    for some constant vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. The vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ are not quite eigenvectors and they are a bit more difficult to find.

Wed, Apr 19

- A system of two linear first order differential equations has the form

$$\begin{cases} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t) \end{cases} \quad \rightsquigarrow \quad \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \rightsquigarrow \quad \mathbf{x}'(t) = A\mathbf{x}(t)$$

- The solution can be easily stated:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp(At) \begin{pmatrix} x(0) \\ y(0) \end{pmatrix},$$

where $\exp(At)$ is the "matrix exponential", which your computer knows how to find.

- For example, my computer says that

$$\exp\begin{pmatrix}0 & -t\\ t & 0\end{pmatrix} = \begin{pmatrix}\cos t & -\sin t\\ \cos t & \sin t\end{pmatrix}.$$

Hence the linear system

$$\begin{cases} x'(t) &= -y(t)\\ y'(t) &= x(t)\end{cases} \quad\rightsquigarrow\quad \begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

has solution

$$\begin{aligned}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} &= \exp\begin{pmatrix}0 & -t\\ t & 0\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}\\ &= \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}\\ &= \begin{pmatrix}x(0)\cos t - y(0)\sin t\\ x(0)\sin t + y(0)\cos t\end{pmatrix}.\end{aligned}$$

- This is the quickest way to solve the problem using a computer. When working by hand we will instead by the eigenvalues and eigenvectors of the matrix A. The matrix equation

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v}\\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} &= \lambda\begin{pmatrix}u\\ v\end{pmatrix}\\ \begin{pmatrix}au+by\\ cu+dy\end{pmatrix} &= \begin{pmatrix}\lambda u\\ \lambda v\end{pmatrix}\end{aligned}$$

becomes

$$\begin{cases} au+by &= \lambda u\\ cu+dv &= \lambda v\end{cases}$$

$$\begin{cases} (a-\lambda)u+by &= 0\\ cu+(d-\lambda)v &= 0\end{cases}$$

If λ is randomly chosen then the only solution for u and v is $u = v = 0$. To get an interesting solution the number λ must satisfy the *characteristic equation*:

$$\begin{aligned}\begin{vmatrix}a-\lambda & b\\ c & d-\lambda\end{vmatrix} &= 0\\ (a-\lambda)(d-\lambda)-bc &= 0\\ \lambda^2-(a+d)\lambda+(ad-bc) &= 0\end{aligned}$$

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}.$$

These two numbers are called the *eigenvalues* of the matrix A. Having found the eigenvalues we back substitute to find the corresponding *eigenvectors* $\mathbf{v} = (u, v)$.

- Example: The eigenvector equation

$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

leads to the characteristic equation

$$\begin{aligned} \begin{vmatrix} -3 - \lambda & 1 \\ 2 & -2 - \lambda \end{vmatrix} &= 0 \\ (-3 - \lambda)(-2 - \lambda) - 1 \cdot 2 &= 0 \\ \lambda^2 + 5\lambda + 4 &= 0 \\ (\lambda + 1)(\lambda + 4) &= 0 \\ \lambda &= -1, -4. \end{aligned}$$

- The eigenvectors corresponding to $\lambda = -1$ are

$$\begin{cases} (-3 - \lambda)u - v &= 0 \\ 2u + (-2 - \lambda)v &= 0 \end{cases}$$

$$\begin{cases} (-3 - (-1))u + v &= 0 \\ 2u + (-2 - (-1))v &= 0 \end{cases}$$

$$\begin{cases} -2u + v &= 0 \\ 2u - v &= 0 \end{cases}$$

There are infinitely many solutions. Let's choose $(u, v) = (1, 2)$. We verify that

$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- The eigenvectors corresponding to $\lambda = -4$ are

$$\begin{cases} (-3 - \lambda)u - v &= 0 \\ 2u + (-2 - \lambda)v &= 0 \end{cases}$$

$$\begin{cases} (-3 - (-4))u + v &= 0 \\ 2u + (-2 - (-4))v &= 0 \end{cases}$$

$$\begin{cases} u + v &= 0 \\ 2u + 2v &= 0 \end{cases}$$

There are infinitely many solutions. Let's choose $(u, v) = (1, -1)$. We verify that

$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Once we have found the eigenvalues and eigenvectors of a matrix, we can solve any linear system of ODEs involving this matrix. Suppose that $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ with $\lambda_1 \neq \lambda_2$. Then the first order linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ has general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ae^{\lambda_1 t}\mathbf{v}_1 + be^{\lambda_2 t}\mathbf{v}_2.$$

For the matrix in the previous example, the first order linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ has general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ae^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + be^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} ae^{-t} + be^{-4t} \\ 2ae^{-t} - be^{-4t} \end{pmatrix}.$$

The parameters a, b are determined by the initial conditions $x(0)$, $y(0)$.

- We can solve **second order linear systems** using the same method. Consider a second order system

$$\begin{cases} x''(t) &= ax(t) + by(t) \\ y''(t) &= cx(t) + dy(t) \end{cases} \quad \rightsquigarrow \quad \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \rightsquigarrow \quad \mathbf{x}''(t) = A\mathbf{x}(t).$$

This represents a pair of coupled oscillators. In a physically realistic problem the two eigenvalues of A will be negative, so we can write them as $\lambda_1 = -\omega_1^2$ and $\lambda_2 = -\omega_2^2$ for some positive real numbers $\omega_1 > 0$ and $\omega_2 > 0$. If $A\mathbf{v}_1 = -\omega_1^2\mathbf{v}_1$ and $A\mathbf{v}_2 = -\omega_2^2\mathbf{v}_2$ then the general solution is

$$\mathbf{x}(t) = (a_1\cos(\omega_1 t) + b_1\sin(\omega_1 t)\mathbf{v}_1 + (a_2\cos(\omega_2 t) + b_2\sin(\omega_2 t)\mathbf{v}_2.$$

The matrix in our previous example, happens to have negative eigenvalues $\lambda_1, \lambda_2 = -1, -4$ so we can take $\omega_1 = 1$, $\omega_2 = 2$. Thus the second order system

$$\begin{cases} x''(t) &= -3x(t) + y(t) \\ y''(t) &= 2x(t) - 2y(t) \end{cases}$$

has general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (a_1\cos(t) + b_1\sin(t))\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2\cos(2t) + b_2\sin(2t))\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The four parameters a_1, b_1, a_2, b_2 are determined by four initial conditions $x(0)$, $x'(0)$, $y(0)$, $y'(0)$. See the homework for another example.

- Warning: Repeated eigenvalues lead to complications that we don't have time to discuss.

Fri, Apr 21

- General Story: A first order linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ has basic solutions of the form $\mathbf{v}e^{\lambda t}$ where $\mathbf{v}$ is a constant vector and λ is a constant satisfying $A\mathbf{v} = \lambda\mathbf{v}$. The general solution is a linear combination of basic solutions:[5]

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_ne^{\lambda_n t}.$$

  The n constants $c_1, \ldots, c_n$ are determined by the n initial conditions $x_1(0), \ldots, x_n(0)$. In our case we only looked at 2×2 systems, which have general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}.$$

- We also considered second order systems $\mathbf{x}''(t) = K\mathbf{x}(t)$, where K is a matrix whose eigenvalues are **negative**. Suppose the eigenvalues are $\lambda_1, \ldots, \lambda_n$ where $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ and $\lambda_k = -\omega_k^2$. Then the system has $2n$ basic solutions:

$$\mathbf{v}_1\cos(\omega_1 t), \quad \mathbf{v}_1\sin(\omega_1 t), \quad \cdots, \quad \mathbf{v}_n\cos(\omega_n t), \quad \mathbf{v}_n\sin(\omega_n t).$$

  And the general solution is a combination of these:

$$\mathbf{x}(t) = \sum_{k=1}^{n}(a_k\mathbf{v}_k\cos(\omega_k t) + b_k\mathbf{v}_k\sin(\omega_k t)).$$

  The $2n$ constants $a_1, b_1, \ldots, a_n, b_n$ are determined by the $2n$ initial conditions $x_1(0), \ldots, x_n(0)$ and $x_1'(0), \ldots, x_n'(0)$.

- Recall the example from last time:

$$\begin{cases} x''(t) &= -3x(t) + y(t) \\ y''(t) &= 2x(t) - 2y(t) \end{cases} \quad \rightsquigarrow \quad \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \rightsquigarrow \quad \mathbf{x}''(t) = A\mathbf{x}(t).$$

  We found the eigenvalues $\lambda_1, \lambda_2 = -1, -4$ and the eigenvectors $\mathbf{v}_1 = (1, 2)$, $\mathbf{v}_2 = (1, -1)$. Check:

$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3+2 \\ 2-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3-1 \\ 2+2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = -4\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

  Note that the eigenvalues are negative: $-1 = -1^2$ and $-4 = -2^2$. Thus we have four basic solutions:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos(t), \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin(t), \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2t), \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(2t).$$

  And the general solution is

$$\mathbf{x}(t) = a_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos(t) + b_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin(t) + a_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2t) + b_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(2t).$$

[5] We assume that the eigenvalues are distinct. Repeated eigenvalues lead to slight complications.

- The four constants a_1, b_1, a_2, b_2 will be determined by the four initial conditions $x(0)$, $x'(0)$, $y(0)$, $y'(0)$. For example, let's take

$$\mathbf{x}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(0) = \begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Substituting $t = 0$ into the solution gives the system

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{cases} 0 &= a_1 + a_2, \\ 1 &= 2a_1 - a_2, \end{cases}$$

which has solution $a_1 = 1/3$ and $a_2 = -1/3$. Substituting $t = 0$ into $\mathbf{x}'(t)$ gives system

$$\begin{aligned}
\mathbf{x}'(t) &= -a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin(t) + b_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(t) - 2a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t) + 2b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t), \\
\mathbf{x}'(0) &= -a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} 0 + b_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} 1 - 2a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} 0 + 2b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} 1, \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= b_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
&\rightsquigarrow \quad \begin{cases} 0 &= b_1 + 2b_2, \\ 1 &= 2b_1 - 2b_2, \end{cases}
\end{aligned}$$

which has solution $b_1 = 1/3$ and $b_2 = -1/6$. We conclude that

$$\mathbf{x}(t) = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(t) + \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin(t) - \frac{1}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) - \frac{1}{6} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t).$$

Equivalently, we have

$$x(t) = \frac{1}{3} \cos(t) + \frac{1}{3} \sin(t) - \frac{1}{3} \cos(2t) - \frac{1}{6} \sin(2t)$$

and

$$y(t) = \frac{2}{3} \cos(t) + \frac{2}{3} \sin(t) + \frac{1}{3} \cos(2t) + \frac{1}{6} \sin(2t).$$

- The big picture: The most general (interesting) linear system is

$$M\mathbf{x}''(t) + G\mathbf{x}'(t) + K\mathbf{x}(t) = \mathbf{f}(t).$$

This represents a system of interacting components with inertia (mass) matrix M, diffusion (friction) matrix G, stiffness (spring) matrix K, and input functions (external forces) $\mathbf{f}(t)$. The method for solving this is based on the same ideas that we just discussed. However, it must be solved with a computer.

- This theory is the basis of most scientific computation. Continuous processes are discretized and nonlinear systems are linearized.

Mon, Apr 24

- We discussed the Homework 6 solutions. Please read them: `https://www.math.miami.edu/~armstrong/311sp23/311sp23hw6sol.pdf`
- We also reviewed for Exam 2. Here are the practice problems: `https://www.math.miami.edu/~armstrong/311sp23/311sp23exam2practice.pdf`