Math 311 **Spring 2023**
Homework 6 **Drew Armstrong**

Given a square matrix A, a constant λ and a vector $\mathbf{v}$ satisfying

$$A\mathbf{v} = \lambda\mathbf{v},$$

we say that $\mathbf{v}$ *is an eigenvector of A with eigenvalue λ*. The eigenvalues of a given matrix are the roots of its *characteristic polynomial*. In the case of a 2×2 matrix we have

$$\begin{array}{ccc} \text{(matrix)} & \rightsquigarrow & \text{(characteristic polynomial)} \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \rightsquigarrow & \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0. \end{array}$$

Once an eigenvalue λ is found, the corresponding eigenvectors $\mathbf{v} = (u, v)$ can be found by solving a linear system:

$$A\mathbf{v} = \lambda\mathbf{v} \quad \rightsquigarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \quad \rightsquigarrow \quad \begin{cases} (a-\lambda)u + bv &= 0, \\ cu + (d-\lambda)v &= 0. \end{cases}$$

Given the matrix A we may consider the system of first order linear differential equations:

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad \rightsquigarrow \quad \begin{cases} x'(t) &= ax(t) + by(t), \\ y'(t) &= cx(t) + dy(t). \end{cases}$$

If the matrix A has distinct eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then the general solution of the system is

$$\mathbf{x}(t) = a_1 e^{\lambda_1 t}\mathbf{v}_1 + a_2 e^{\lambda_2 t}\mathbf{v}_2.$$

We may also consider the system of second order differential equations:

$$\mathbf{x}''(t) = A\mathbf{x}(t) \quad \rightsquigarrow \quad \begin{cases} x''(t) &= ax(t) + by(t), \\ y''(t) &= cx(t) + dy(t). \end{cases}$$

If the eigenvalues are negative, say $\lambda_1 = -\omega_1^2$ and $\lambda_2 = -\omega_2^2$, then the general solution of the second order system is

$$\mathbf{x}(t) = (a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t))\mathbf{v}_1 + (a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t))\mathbf{v}_2.$$

1. First Order Linear System. Consider the following system of differential equations:

$$\begin{cases} 5x'(t) &= x(t) + 6y(t), \\ 5y'(t) &= 4x(t) - y(t). \end{cases}$$

(a) Find the 2×2 matrix A such that $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x(t), y(t))$.
(b) Find the eigenvalues and eigenvectors of A.
(c) Use part (b) to find the general solution of the system.
(d) Find the specific solution with initial conditions $x(0) = 0$ and $y(0) = 5$.

2. Second Order Linear System. Consider the following system of differential equations:

$$\begin{cases} x''(t) &= -2x(t) + 2y(t), \\ y''(t) &= x(t) - 3y(t). \end{cases}$$

(a) Find the 2×2 matrix A such that $\mathbf{x}''(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x(t), y(t))$.
(b) Find the eigenvalues and eigenvectors of A.
(c) Use (b) to find the general solution of the system.
(d) Find the specific solution with initial conditions $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$.