Math 311 **Spring 2023**
Homework 5 **Drew Armstrong**

The *Laplace transform* $F(s)$ of a function $f(t)$ is defined as follows:

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st}\,dt.$$

One can use this definition to derive the following general rules:

(1) $\mathscr{L}[t \cdot f(t)] = -F'(s)$
(2) $\mathscr{L}[e^{at} \cdot f(t)] = F(s-a)$
(3) $\mathscr{L}[f'(t)] = sF(s) - f(0)$
(4) $\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$
(5) $\mathscr{L}[H(t-a) \cdot f(t-a)] = e^{-as}F(s)$, where $H(t)$ is the *Heaviside step function*:

$$H(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 1. \end{cases}$$

Here are the transforms of some basic functions:

- $\mathscr{L}[0] = 0$
- $\mathscr{L}[1] = 1/s$
- $\mathscr{L}[e^{at}] = 1/(s-a)$
- $\mathscr{L}[t] = 1/s^2$
- $\mathscr{L}[t^n] = n!/s^{n+1}$
- $\mathscr{L}[\cos(kt)] = s/(s^2+k^2)$
- $\mathscr{L}[\sin(kt)] = k/(s^2+k^2)$

The *Dirac delta function* $\delta(t)$ satisfies $\mathscr{L}[\delta(t-a)] = e^{-as}$.

1. Using the Rules.

(a) Use rule (1) to compute

$$\mathscr{L}\left[t \cdot \sin(kt)\right] \quad \text{and} \quad \mathscr{L}\left[t \cdot \cos(kt)\right].$$

(b) Use rule (2) to compute

$$\mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right] \quad \text{and} \quad \mathscr{L}^{-1}\left[\frac{2}{(s-3)^2+4}\right].$$

(c) Use rule (5) to compute

$$\mathscr{L}^{-1}\left[\frac{e^{-s}}{s}\right] \quad \text{and} \quad \mathscr{L}^{-1}\left[\frac{e^{-2s}}{s^2+1}\right].$$

2. Some Small Problems. Solve using Laplace transforms:

(a) $x'(t) = x(t)$; $x(0) = 1$
(b) $x'(t) = x(t) + 1$; $x(0) = 1$
(c) $x'(t) = x(t) + e^t$; $x(0) = 1$

3. A Bigger Problem.

(a) Find the partial fraction expansion of $\frac{1}{(s-2)(s-3)}$.
(b) Find the partial fraction expansion of $\frac{s}{(s-2)(s-3)}$.
(c) Find the partial fraction expansion of $\frac{1}{s(s-2)(s-3)}$.

(d) Use Laplace transforms together with (a), (b), (c) to solve the initial value problem:

$$x''(t) - 5x'(t) + 6x(t) = 1; \quad x(0) = 5, \quad x'(0) = 7.$$

4. Resonance. Consider the following initial value problem:

$$x''(t) + 4x(t) = \cos(\omega t); \quad x(0) = x'(0) = 0.$$

(a) First suppose that $\omega \neq 2$. In this case solve for A, B, C, D in the expansion:

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2}.$$

(b) Use part (a) and Laplace transforms to solve the initial value problem when $\omega \neq 2$.
(c) Use Problem 1(a) and Laplace transforms to solve the initial value problem when $\omega = 2$.

5. Hitting a Spring with a Hammer. The undamped oscillator $x''(t) + x(t) = 0$ with initial conditions $x(0) = 0$ and $x'(0) = 1$ has solution $x(t) = \sin t$. If we hit this spring with a hammer at time $t = a > 0$, then the equation becomes

$$x''(t) + x(t) = \delta(t-a); \quad x(0) = 0, x'(0) = 1.$$

(a) Solve the new equation. [Hint: Use rule (5). Your answer will involve $H(t-a)$.]
(b) Use a computer to graph your solution for the following three values of a:

$$a = \frac{9\pi}{10}, \qquad a = \pi, \qquad a = \frac{11\pi}{10}.$$

6. Hockey Puck on Ice. A hockey puck of mass $m = 1$ sits on a flat sheet of ice with friction $\gamma > 0$. At time $t = 0$ a hockey stick instantaneously transfers 1 Newton of force to the puck. Let $x(t)$ be the horizontal distance of the puck from the hockey player, so that

$$x''(t) + \gamma \cdot x'(t) = \delta(t); \quad x(0) = x'(0) = 0.$$

(a) Find the partial fraction expansion of $\frac{1}{s(s+\gamma)}$.
(b) Solve for $x(t)$ in terms of γ. [Hint: Your answer will involve $H(t)$.]
(c) How far does the puck go before it is stopped by friction? [Hint: $\lim_{t\to\infty} x(t)$.]