Math 311 **Spring 2023**
Homework 4 **Drew Armstrong**

A *linear operator* L sends each function $y(x)$ to a function $L[y(x)]$, and satisfies two properties:

- $L[Cy(x)] = CL[y(x)]$ for all constants C and functions $y(x)$,
- $L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$ for all functions $y_1(x)$ and $y_2(x)$.

We can also phrase these two properties as one property:

- $L[C_1y_1(x) + C_2y_2(x)] = C_1L[y_1(x)] + C_2L[y_2(x)]$ for all constants C_1, C_2 and functions $y_1(x), y_2(x)$.

A *linear differential operator* has the form

$$L[y(x)] = P_0(x)y(x) + P_1(x)y'(x) + P_2(x)y''(x) + \cdots + P_ny^{(n)}(x)$$

for some functions $P_0(x), \ldots, P_n(x)$. A *linear ODE* has the form $L[y(x)] = f(x)$, where L is a linear differential operator and $f(x)$ is any function. The general solution of the linear ODE is $y(x) = y_c(x) + y_p(x)$, where

- $y_c(x)$ is the general solution of the *homogeneous* equation $L[y(x)] = 0$,
- $y_p(x)$ is any one particular solution of the *non-homogeneous* equation $L[y(x)] = f(x)$.

1. Linear Operators. Test whether each of the following operators is linear:

(a) $L[y(x)] = y'(x)$
(b) $L[y(x)] = y(x)^2$
(c) $L[y(x)] = y'(x) \cdot y(x)$
(d) $L[y(x)] = \int_0^x y(s)\,ds$

(a): Consider any constants C_1, C_2 and functions $y_1(x), y_2(x)$. We note that

$$C_1L[y_1(x)] + C_2L[y_2(x)] = C_1y_1'(x) + C_2y_2'(x)$$

and

$$\begin{aligned} L[C_1y_1(x) + C_2y_2(x)] &= [C_1y_1(x) + C_2y_2(x)]' \\ &= C_1y_1'(x) + C_2y_2'(x). && \text{property of derivatives} \end{aligned}$$

Since these are the same, we conclude that L is **linear**.

(b): Consider any constants C_1, C_2 and functions $y_1(x), y_2(x)$. We note that

$$C_1L[y_1(x)] + C_2L[y_2(x)] = C_1y_1(x)^2 + C_2y_2(x)^2$$

and

$$\begin{aligned} L[C_1y_1(x) + C_2y_2(x)] &= [C_1y_1(x) + C_2y_2(x)]^2 \\ &= C_1^2y_1(x)^2 + C_2^2y_2(x)^2 + 2C_1C_2y_1(x)y_2(x). \end{aligned}$$

Since these are not the same, we conclude that L is **not linear**.

(b): Consider any constants C_1, C_2 and functions $y_1(x), y_2(x)$. We note that

$$C_1L[y_1(x)] + C_2L[y_2(x)] = C_1y_1'(x)y_1(x) + C_2y_2'(x)y_2(x)$$

and

$$\begin{aligned} L[C_1y_1(x) + C_2y_2(x)] &= [C_1y_1(x)] + C_2y_2(x)]' \cdot [C_1y_1(x) + C_2y_2(x)] \\ &= [C_1y_1'(x) + C_2y_2'(x)] \cdot [C_1y_1(x) + C_2y_2(x)] \end{aligned}$$

$$= C_1^2 y_1'(x) y_1(x) + C_2^2 y_2'(x) y_2(x) + C_1 C_2 y_1'(x) y_2(x) + C_1 C_2 y_1(x) y_2'(x).$$

Since these are not the same, we conclude that L is **not linear**.

(a): Consider any constants C_1, C_2 and functions $y_1(x), y_2(x)$. We note that

$$C_1 L[y_1(x)] + C_2 L[y_2(x)] = C_1 \int_0^x y_1(s)\, ds + C_2 \int_0^x y_2(s)\, ds$$

and

$$\begin{aligned} L[C_1 y_1(x) + C_2 y_2(x)] &= \int_0^x [C_1 y_1(s) + C_2 y_2(s)]\, ds \\ &= C_1 \int_0^x y_1(s)\, ds + C_2 \int_0^x y_2(s)\, ds. \qquad \text{property of integrals} \end{aligned}$$

Since these are the same, we conclude that L is **linear**.

Remark: In this problem we observed that differentiation and integration are linear operators. This is the reason why linear operators are important in the study of differential equations.

2. Undetermined Coefficients I. The *method of undetermined coefficients* uses an educated guess to find one particular solution of a non-homogeneous linear ODE:

(a) Find one solution to $x'(t) + x(t) = 5$. [Hint: Guess $x_p(t) = A$.]
(b) Find one solution to $x'(t) + x(t) = t^2$. [Hint: Guess $x_p(t) = At^2 + Bt + C$.]
(c) Find one solution to $x'(t) + x(t) = \cos t$. [Hint: Guess $x_p(t) = A\cos t + B \sin t$.]

(a): We substitute the guess $x_p(t) = A$ to obtain

$$\begin{aligned} x_p'(t) + x_p(t) &= 5 \\ (A)' + A &= 5 \\ 0 + A &= 5 \\ A &= 5. \end{aligned}$$

Hence $\boxed{x_p(t) = 5}$ is a solution.

(b): We substitute the guess $x_p(t) = At^2 + Bt + C$ to obtain

$$\begin{aligned} x_p'(t) + x_p(t) &= t^2 \\ (At^2 + Bt + C)' + At^2 + Bt + C &= t^2 \\ (2At + B) + At^2 + Bt + C &= t^2 \\ (A)t^2 + (2A + B)t + (B + C) &= 1t^2 + 0t + 0. \end{aligned}$$

Since we have 3 unknowns A, B, C we need 3 equations. Actually we have infinitely many equations, one for each value of t. To get 3 equations we can substitute any 3 values of t. Another method is just to compare coefficients in the polynomial expansions:[1]

$$\begin{cases} A &= 1 \\ 2A + B &= 0 \\ B + C &= 0. \end{cases}$$

[1] Technically: The method of comparing coefficients is the same as (1) putting $t = 0$, (2) differentiating and putting $t = 0$, (3) differentiating twice and putting $t = 0$.

Solving this system gives $A = 1$, $B = -2$ and $C = 2$, hence

$$\boxed{x_p(t) = t^2 - 2t + 2.}$$

(c): We substitute the guess $x_p(t) = A\cos t + B\sin t$ to obtain

$$\begin{aligned} x_p'(t) + x_p(t) &= \cos t \\ (A\cos t + B\sin t)' + A\cos t + B\sin t &= \cos t \\ -A\sin t + B\cos t + A\cos t + B\sin t &= \cos t \\ (A+B)\cos t + (-A+B)\sin t &= 1\cos t + 0\sin t. \end{aligned}$$

Since we have two unknowns A, B we need two equations. Actually we have infinitely many equations, one for each value of t. To get two equations we can substitute any two values of t. Another method is just to compare coefficients:[2]

$$\begin{cases} A + B &= 1 \\ -A + B &= 0. \end{cases}$$

Solving this system gives $A = 1/2$ and $B = 1/2$, hence

$$\boxed{x_p(t) = \frac{1}{2}\cos t + \frac{1}{2}\sin t.}$$

3. Undetermined Coefficients II. Use your answers from Problem 2 to solve the following initial value problems:

(a) $x'(t) + x(t) = 5$; $x(0) = 0$
(b) $x'(t) + x(t) = t^2$; $x(0) = 0$
(c) $x'(t) + x(t) = \cos t$; $x(0) = 0$

First we note that the homogeneous linear equation $x'(t) + x(t) = 0$ has general solution

$$x_c(t) = Ce^{-t}.$$

(a): The general solution of $x'(t) + x(t) = 5$ is

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) \\ &= Ce^{-t} + 5. \end{aligned}$$

To find C we substitute $x(0) = 0$:

$$\begin{aligned} x(0) &= 0 \\ Ce^0 + 5 &= 0 \\ C + 5 &= 0 \\ C &= -5. \end{aligned}$$

Hence the solution is

$$\boxed{x(t) = -5e^{-t} + 5.}$$

(a): The general solution of $x'(t) + x(t) = t^2$ is

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) \\ &= Ce^{-t} + t^2 - 2t + 2. \end{aligned}$$

[2]Technically: The method of comparing coefficients is the same as substituting $t = 0$ and $t = \pi/2$.

To find C we substitute $x(0) = 0$:

$$\begin{aligned} x(0) &= 0 \\ Ce^0 + 0 - 0 + 2 &= 0 \\ C + 2 &= 0 \\ C &= -2. \end{aligned}$$

Hence the solution is

$$\boxed{x(t) = -2e^{-t} + t^2 - 2t + 2.}$$

(a): The general solution of $x'(t) + x(t) = \cos t$ is

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) \\ &= Ce^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t. \end{aligned}$$

To find C we substitute $x(0) = 0$:

$$\begin{aligned} x(0) &= 0 \\ Ce^0 + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 &= 0 \\ C + \frac{1}{2} &= 0 \\ C &= -\frac{1}{2}. \end{aligned}$$

Hence the solution is

$$\boxed{x(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t.}$$

4. The Hanging Spring. Consider a particle of mass $m > 0$ hanging from the ceiling by a (massless, frictionless) spring with stiffness $k > 0$. Let $y(t)$ be the height of the mass at time t. Let $y = 0$ be the bottom of the spring **when the mass is not attached**, so the spring force is $-ky(t)$. Then $y(t)$ satisfies the differential equation

$$\begin{aligned} (\text{force}) &= (\text{spring}) + (\text{gravity}) \\ my''(t) &= -ky(t) - gm \\ my''(t) + ky(t) &= -gm, \end{aligned}$$

where $g > 0$ is the gravitational constant. This is a linear ODE. **Find the general solution**. [Hint: First find the general homogeneous solution $y_c(t)$. Then find a particular solution $y_p(t)$. Since the non-homogeneous term $-gm$ is constant, look for a constant solution $y_p(t) = A$.]

This is a non-homogeneous linear equation. Consider the homogeneous equation:

$$my''(t) + ky(t) = 0.$$

We have solved this equation before. Recall: It has general solution $y_c(t) = c_1e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, where λ_1, λ_2 are the two roots of the characteristic equation $m\lambda^2 + k = 0$. Let's write $\omega = \sqrt{k/m} > 0$. Since the roots are $\lambda_1, \lambda_2 = \pm\sqrt{-k/m} = \pm i\omega$ we conclude that

$$\begin{aligned} y_c(t) &= c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ &= c_3\cos(\omega t) + c_4\sin(\omega t) \end{aligned}$$

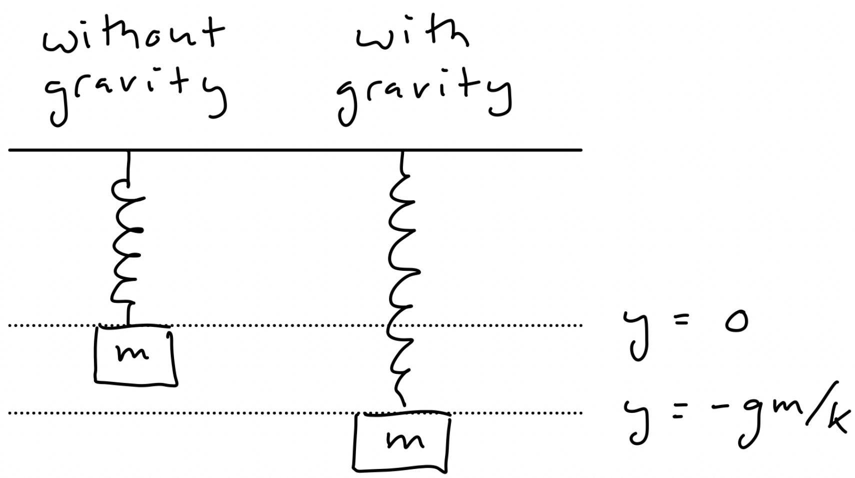
for some constants c_3, c_4. Now we consider the non-homogeneous equation $my''(t) + ky(t) = -gm$. Since the right hand side is constant, we guess that there is a constant solution $y_p(t) = A$. Substituting gives

$$\begin{aligned} my_p''(t) + ky_p(t) &= -gm \\ m(A)'' + kA &= -gm \\ 0 + kA &= -gm \\ A &= -gm/k. \end{aligned}$$

Hence the general solution of $my''(t) + ky(t) = -gm$ is

$$\boxed{y(t) = y_c(t) + y_p(t) = c_3 \cos(\sqrt{k/m} \cdot t) + c_4 \sin(\sqrt{k/m} \cdot t) - gm/k.}$$

Interpretation: Without gravity, the equilibrium height is $y = 0$. With gravity, the new equilibrium height is $y = -gm/k$. The frequency of oscillation doesn't change. Of course, this is an idealized situation where the spring is massless, there is no friction, and the oscillations are small enough that $-ky(t)$ is a realistic model of the spring force. Picture:



5. Variation of Parameters. The method of undetermined coefficients only works sometimes. The method of *variation of parameters* always works, but the computations are usually more difficult.

(a) The homogeneous equation $y'(x) + y(x) = 0$ has general solution $y_c(x) = Ce^{-x}$. So the non-homogeneous equation $y'(x) + y(x) = x^2$ has a solution $y_p(x) = u(x)e^{-x}$ for some function $u(x)$.[3] Substitute this into the ODE and solve for $u(x)$.

(b) The homogeneous equation $y'(x) - y(x)/x = 0$ has general solution $y_c(x) = Cx$, so the non-homogeneous equation $y'(x) - y(x)/x = x$ has a solution $y_p(x) = u(x)x$. Substitute and solve for $u(x)$.

(c) The homogeneous equation $x''(t) - 3x'(t) + 2x(t) = 0$ has general solution $x_c(t) = c_1e^t + c_2e^{2t}$, so the non-homogeneous equation $x''(t) - 3x'(t) + 2x(t) = e^{3t}$ has a solution $x_p(t) = u_1(t)e^t + u_2(t)e^{2t}$ for some functions $u_1(t)$ and $u_2(t)$. Substitute and solve for $u_1(t)$ and $u_2(t)$. [Hint: You may assume for simplicity that $u_1'(t)e^t + u_2'(t)e^{2t} = 0$.]

[3] The method is called "variation of paramters" because we turn the parameter into a function.

(a): Substitute the guess $y_p(x) = u(x)e^{-x}$ to obtain

$$\begin{aligned}
y_p'(x) + y(x) &= x^2 \\
[u(x)e^{-x}]' + y(x) &= x^2 \\
u'(x)e^{-x} - \cancel{u(x)e^{-x}} + \cancel{u(x)e^{-x}} &= x^2 \\
u'(x)e^{-x} &= x^2 \\
u'(x) &= x^2e^x \\
u(x) &= \int x^2e^x\, dx.
\end{aligned}$$

We can solve this using integration by parts:

$$\begin{aligned}
\int x^2e^x\, dx &= \int udv && (u = x^2, dv = e^x dx) \\
&= uv - \int vdu \\
&= x^2e^x - \int 2xe^x\, dx \\
&= x^2e^x - 2\int wdz && (w = x, dz = e^x dx) \\
&= x^2e^x - 2\left(wz - \int zdw\right) \\
&= x^2e^x - 2\left(xe^x - \int e^x dx\right) \\
&= x^2e^x - 2\left(xe^x - e^x\right) \\
&= (x^2 - 2x + 2)e^x.
\end{aligned}$$

Hence we obtain a particular solution

$$\boxed{y_p(x) = u(x)e^{-x} = (x^2 - 2x + 2)e^x \cdot e^{-x} = x^2 - 2x + 2.}$$

Wait a minute! We already found this solution in Problem 2(b). Which method was easier?

(b): I apologize that this problem originally had a typo. I will solve the fixed version. The homogeneous linear equation $y'(x) - y(x)/x = 0$ has general solution $y(x) = Cx$. Since this was the typo, let me check that this is correct. I will use separation of variables:

$$\begin{aligned}
\frac{dy}{dx} - \frac{y}{x} &= 0 \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{y} &= \frac{dx}{x} \\
\int \frac{dy}{y} &= \int \frac{dx}{x} + B \\
\ln(y) &= \ln(x) + B \\
y &= xe^B \\
y &= Cx.
\end{aligned}$$

Okay, it's correct. Based on this we should have a particular solution of the form $y_p(x) = u(x) \cdot x$. Substituting gives

$$\begin{aligned} y_p'(x) + y_p(x)/x &= x \\ [u(x) \cdot x]' - u(x) &= x \\ u'(x) \cdot x + \cancel{u(x) \cdot 1} - \cancel{u(x)} &= x \\ u'(x) \cdot x &= x \\ u'(x) &= 1 \\ u(x) &= x. \end{aligned}$$

Hence we obtain a particular solution:

$$\boxed{y_p(x) = u(x) \cdot x = x \cdot x = x^2.}$$

(c): The general homogeneous solution is

$$x_c(t) = c_1 e^t + c_2 e^{2t},$$

hence we look for a particular solution of the form

$$x_p(t) = u_1(t)e^t + u_2(t)e^{2t},$$

For simplicity we will assume that $u_1'(t)e^t + u_2'(t)e^{2t} = 0$. (This is a good trick that makes the solution computable by hand.) Substituting gives

$$\begin{aligned} e^{3t} &= x_p''(t) - 3x_p'(t) + 2x_p(t) \\ &= [u_1(t)e^t + u_1(t)2e^{2t} + \cancel{u_1'(t)e^t + u_2'(t)e^{2t}}]' \\ &\quad - 3[u_1(t)e^t + u_1(t)2e^{2t} + \cancel{u_1'(t)e^t + u_2'(t)e^{2t}}] \\ &\quad + 2[u_1(t)e^t + u_2(t)e^{2t}] \\ &= [u_1(t)e^t + u_1(t)2e^{2t}]' \\ &\quad - 3[u_1(t)e^t + u_1(t)2e^{2t}] \\ &\quad + 2[u_1(t)e^t + u_2(t)e^{2t}] \\ &= [u_1'(t)e^t + 2u_2'(t)e^{2t} + u_1(t)e^t + 4u_2(t)e^{2t}] \\ &\quad - 3[u_1(t)e^t + u_1(t)2e^{2t}] \\ &\quad + 2[u_1(t)e^t + u_2(t)e^{2t}] \\ &= u_1'(t)e^t + 2u_2'(t)e^{2t} + (1 - 3 + 2)u_1(t)e^t + (4 - 6 + 2)u_2(t)e^{2t} \\ &= u_1'(t)e^t + 2u_2'(t)e^{2t} + 0. \end{aligned}$$

Thus we have two equations:[4]

$$\begin{cases} u_1'(t)e^t + u_2'(t)e^{2t} &= 0, \\ u_1'(t)e^t + 2u_2'(t)e^{2t} &= e^{3t}. \end{cases}$$

(You are free to skip the derivation and go right to the system of two equations.) Subtracting the equations gives

$$u_2'(t)e^{2t} = e^{3t}$$

[4] In general, if $x_1(t)$ and $x_2(t)$ are the homogeneous solutions, then $u_1(t)$ and $u_2(t)$ satisfy the two equations $u_1'(t)x_1(t) + u_2'(t)x_2(t) = 0$ and $u_1'(t)x_1'(t) + u_2'(t)x_2'(t) = f(t)$, where $f(t)$ is the non-homogeneous term.

$$\begin{aligned} u_2'(t) &= e^t \\ u_2(t) &= e^t. \end{aligned}$$

Then substituting into the first equation gives

$$\begin{aligned} u_1'(t)e^t + u_2'(t)e^{2t} &= 0 \\ u_1'(t)e^t + e^t e^{2t} &= 0 \\ u_1'(t)e^t &= -e^{3t} \\ u_1'(t) &= -e^{2t} \\ u_1(t) &= -\frac{1}{2}e^{2t}. \end{aligned}$$

Hence we obtain a particular solution:

$$\boxed{x_p(t) = u_1(t)e^t + u_2(t)e^{2t} = -\frac{1}{2}e^{2t} + e^t e^{2t} = \frac{1}{2}e^{3t}.}$$

Remark: Again, it is easier to find this solution using the method of undetermined coefficients. But apparently the method of variation of parameters is usually taught in this course. Since this is my first time teaching the course I am trying to follow tradition.

6. Beats. Consider a free undamped oscillator with mass $m = 1$ and stiffness $k = 3025$, which satisfies the differential equation

$$x''(t) + 3025x(t) = 0.$$

The natural frequency is $\omega_0 = \sqrt{k/m} = 55$ and the general solution is $x_c(t) = c_1 \cos(55t) + c_2 \sin(55t)$. Now suppose we subject this oscillator to a periodic external force with amplitude 500 and frequency 45:

$$x''(t) + 3025x(t) = 500\cos(45t).$$

(a) Find a particular solution of the form $x_p(t) = A\cos(45t) + B\sin(45t)$.
(b) Find the general solution $x(t) = x_c(t) + x_p(t)$.
(c) Find the unique solution $x(t)$ with initial conditions $x(0) = 0$ and $x'(0) = 0$.
(d) Express your solution in the form $x(t) = C\sin(\alpha t)\sin(\beta t)$. [Hint: Use the trig identities

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta, \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \cos(\alpha - \beta) - \cos(\alpha + \beta) &= 2\sin\alpha\sin\beta.] \end{aligned}$$

(e) Plot your solution $x(t)$ for t between 0 and $3\pi/5$. [Use a computer.]

(a): Substitute the guess $x_p(t) = A\cos(45t) + B\sin(45t)$ to get

$$\begin{aligned} 500\cos(45t) &= x_p''(t) + 3025x_p(t) \\ &= -45^2 A\cos(45t) - 45^2 B\sin(45t) + 3025A\cos(45t) + 3025\sin(45t) \\ &= (3025 - 45^2)A\cos(45t) + (3025 - 45^2)B\sin(45t) \\ &= 1000A\cos(45t) + 1000B\sin(45t). \end{aligned}$$

Comparing coefficients of $\cos(45t)$ and $\sin(45t)$ gives $500 = 1000A$ and $0 = 1000B$, hence $A = 1/2$ and $B = 0$. Thus we obtain the particular solution

$$\boxed{x_p(t) = \frac{1}{2}\cos(45t).}$$

(b): The general solution is

$$\boxed{x(t) = x_c(t) + x_p(t) = c_1 \cos(55t) + c_2 \sin(55t) + \frac{1}{2}\cos(45t).}$$

(c): Substitute the initial condition $x(0) = 0$ to get

$$\begin{aligned} 0 &= x(0) \\ 0 &= c_1 \cos(0) + c_2 \sin(0) + \frac{1}{2}\cos(0) \\ 0 &= c_1 + \frac{1}{2} \\ c_1 &= -\frac{1}{2}. \end{aligned}$$

Then compute the derivative and substitute $x'(0) = 0$ to get

$$\begin{aligned} x'(t) &= - - 55c_1 \sin(55t) + 55c_2 \cos(55t) - \frac{1}{2}45\sin(45t) \\ 0 &= -55c_1 \sin(0) + 55c_2 \cos(0) - \frac{1}{2}45\sin(0) \\ 0 &= 55c_2 \\ c_2 &= 0. \end{aligned}$$

We conclude that

$$\boxed{x(t) = -\frac{1}{2}\cos(55t) + \frac{1}{2}\cos(45t).}$$

(d): Putting $\alpha = 50t$ and $\beta = 5t$ in the trig identity gives

$$\begin{aligned} \cos(45t) - \cos(55t) &= \cos(\alpha - \beta) - \cos(\alpha + \beta) \\ &= 2\sin\alpha\sin\beta \\ &= 2\sin(50t)\sin(5t), \end{aligned}$$

and hence

$$\boxed{x(t) = \sin(50t)\sin(5t).}$$

Yes, this problem was reverse-engineered to have a nice solution.

(e): Here is a plot of $x(t)$ for t from 0 to $3\pi/5$: