Math 311 **Spring 2023**
Homework 4 **Drew Armstrong**

A *linear operator* L sends each function $y(x)$ to a function $L[y(x)]$, and satisfies two properties:

- $L[Cy(x)] = CL[y(x)]$ for all constants C and functions $y(x)$,
- $L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$ for all functions $y_1(x)$ and $y_2(x)$.

We can also phrase these two properties as one property:

- $L[C_1y_1(x) + C_2y_2(x)] = C_1L[y_1(x)] + C_2L[y_2(x)]$ for all constants C_1, C_2 and functions $y_1(x), y_2(x)$.

A *linear differential operator* has the form

$$L[y(x)] = P_0(x)y(x) + P_1(x)y'(x) + P_2(x)y''(x) + \cdots + P_n y^{(n)}(x)$$

for some functions $P_0(x), \ldots, P_n(x)$. A *linear ODE* has the form $L[y(x)] = f(x)$, where L is a linear differential operator and $f(x)$ is any function. The general solution of the linear ODE is $y(x) = y_c(x) + y_p(x)$, where

- $y_c(x)$ is the general solution of the *homogeneous* equation $L[y(x)] = 0$,
- $y_p(x)$ is any one particular solution of the *non-homogeneous* equation $L[y(x)] = f(x)$.

1. Linear Operators. Test whether each of the following operators is linear:

(a) $L[y(x)] = y'(x)$
(b) $L[y(x)] = y(x)^2$
(c) $L[y(x)] = y'(x) \cdot y(x)$
(d) $L[y(x)] = \int_0^x y(s)\, ds$

2. Undetermined Coefficients I. The *method of undetermined coefficients* uses an educated guess to find one particular solution of a non-homogeneous linear ODE:

(a) Find one solution to $x'(t) + x(t) = 5$. [Hint: Guess $x_p(t) = A$.]
(b) Find one solution to $x'(t) + x(t) = t^2$. [Hint: Guess $x_p(t) = At^2 + Bt + C$.]
(c) Find one solution to $x'(t) + x(t) = \cos t$. [Hint: Guess $x_p(t) = A\cos t + B\sin t$.]

3. Undetermined Coefficients II. Use your answers from Problem 2 to solve the following initial value problems:

(a) $x'(t) + x(t) = 5$; $x(0) = 0$
(b) $x'(t) + x(t) = t^2$; $x(0) = 0$
(c) $x'(t) + x(t) = \cos t$; $x(0) = 0$

4. The Hanging Spring. Consider a particle of mass $m > 0$ hanging from the ceiling by a (massless, frictionless) spring with stiffness $k > 0$. Let $y(t)$ be the height of the mass at time t. Let $y = 0$ be the bottom of the spring **when the mass is not attached**, so the spring force is $-ky(t)$. Then $y(t)$ satisfies the differential equation

$$\begin{aligned} (\text{force}) &= (\text{spring}) + (\text{gravity}) \\ my''(t) &= -ky(t) - gm \\ my''(t) + ky(t) &= -gm, \end{aligned}$$

where $g > 0$ is the gravitational constant. This is a linear ODE. **Find the general solution**. [Hint: First find the general homogeneous solution $y_c(t)$. Then find a particular solution $y_p(t)$. Since the non-homogeneous term $-gm$ is constant, look for a constant solution $y_p(t) = A$.]

5. Variation of Parameters. The method of undetermined coefficients only works sometimes. The method of *variation of parameters* always works, but the computations are usually more difficult.

(a) The homogeneous equation $y'(x)+y(x)=0$ has general solution $y_c(x)=Ce^{-x}$. So the non-homogeneous equation $y'(x)+y(x)=x^2$ has a solution $y_p(x)=u(x)e^{-x}$ for some function $u(x)$.[1] Substitute this into the ODE and solve for $u(x)$.

(b) The homogeneous equation $y'(x)-y(x)/x=0$ has general solution $y_c(x)=Cx$, so the non-homogeneous equation $y'(x)-y(x)/x=x$ has a solution $y_p(x)=u(x)x$. Substitute and solve for $u(x)$.

(c) The homogeneous equation $x''(t)-3x'(t)+2x(t)=0$ has general solution $x_c(t)=c_1e^t+c_2e^{2t}$, so the non-homogeneous equation $x''(t)-3x'(t)+2x(t)=e^{3t}$ has a solution $x_p(t)=u_1(t)e^t+u_2(t)e^{2t}$ for some functions $u_1(t)$ and $u_2(t)$. Substitute and solve for $u_1(t)$ and $u_2(t)$. [Hint: You may assume for simplicity that $u_1'(t)e^t+u_2'(t)e^{2t}=0$.]

6. Beats. Consider a free undamped oscillator with mass $m=1$ and stiffness $k=3025$, which satisfies the differential equation

$$x''(t)+3025x(t)=0.$$

The natural frequency is $\omega_0=\sqrt{k/m}=55$ and the general solution is $x_c(t)=c_1\cos(55t)+c_2\sin(55t)$. Now suppose we subject this oscillator to a periodic external force with amplitude 500 and frequency 45:

$$x''(t)+3025x(t)=500\cos(45t).$$

(a) Find a particular solution of the form $x_p(t)=A\cos(45t)+B\sin(45t)$.

(b) Find the general solution $x(t)=x_c(t)+x_p(t)$.

(c) Find the unique solution $x(t)$ with initial conditions $x(0)=0$ and $x'(0)=0$.

(d) Express your solution in the form $x(t)=C\sin(\alpha t)\sin(\beta t)$. [Hint: Use the trig identities

$$\begin{aligned}\cos(\alpha-\beta)&=\cos\alpha\cos\beta+\sin\alpha\sin\beta,\\ \cos(\alpha+\beta)&=\cos\alpha\cos\beta-\sin\alpha\sin\beta,\\ \cos(\alpha-\beta)-\cos(\alpha+\beta)&=2\sin\alpha\sin\beta.]\end{aligned}$$

(e) Plot your solution $x(t)$ for t between 0 and $3\pi/5$. [Use a computer.]

[1] The method is called "variation of paramters" because we turn the parameter into a function.