**Math 311** **Spring 2023**
**Homework 2** **Drew Armstrong**

**1. Integrating Factors for Linear ODEs.** Solve the following equations for $y(x)$:

(a) $y' + y = e^x$ and $y(0) = 1$,
(b) $xy' + 2y = 3x$ and $y(1) = 5$,
(c) $xy' - y = x$ and $y(1) = 7$,
(d) $y' = 1 + 2xy$ and $y(0) = 5$. [Express your answer in terms of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds.]$$

**2. Logistic Growth with Harvesting.** Let $x(t)$ be the size of a farmed population (maybe fish in a pond). Without harvesting, let's say the population has logistic growth $x'(t) = x(4-x)$. If we harvest the population at a constant rate $h > 0$ then we obtain the equation

$$x'(t) = x(4-x) - h, \quad \text{where } h > 0 \text{ is the constant rate of harvesting.}$$

Solve the following problems for three different rates of harvesting: $h = 3, 4, 5$.

(a) For which values of $x$ is $x(4-x) - h$ positive, zero, negative?
(b) Use part (a) to sketch the slope field.
(c) Describe the behavior of $x(t)$ as $t \to \infty$. [Ignore negative solutions. If $x(t)$ becomes negative we say that the population is extinct.]

Remark: These equations can be exactly solved, but I'm not asking you to do that.

**3. Phase Shift.** The angle sum identity for cosine tells us that

$$C\cos(x-\alpha) = C\cos\alpha\cos x + C\sin\alpha\sin x.$$

(a) Suppose that $C\cos(x-\alpha) = A\cos x + B\sin x$. Use the above identity to express $C$ and $\alpha$ in terms of $A$ and $B$. [Hint: We must have $A = C\cos\alpha$ and $B = C\sin\alpha$.]
(b) Use part (a) to express $\cos x + \sin x$ in the form $C\cos(x-\alpha)$.
(c) Graph the three functions $\cos x$, $\sin x$ and $C\cos(x-\alpha)$ on the same axes to make sure that your answer in part (b) makes sense.

**4. Indoor vs Outdoor Temperature.** We will use the function $\cos(t)$ to model the outdoor temperature. If $u(t)$ is the indoor temperature then Newton's Law says[1]

$$u'(t) = \cos(t) - u(t).$$

(a) Compute the general solution. [Hint: You will need the integral

$$\int e^t \cos t\, dt = \frac{e^t}{2}(\cos t + \sin t) + C.]$$

(b) Find the specific solution with $u(0) = 3$. Use a computer to graph the indoor temperature $u(t)$ and the outdoor temperature $\cos(t)$ on the same axes, say for $t = 0 \ldots 15$.
(c) As $t \to \infty$ the indoor temperature settles down to a simple oscillation. Compute the phase shift between the indoor and outdoor temperatures. After the outdoor temperature peaks, how many hours until the indoor temperature peaks? [Assume the outdoor temperature has a period of 24 hours.]

[1] Technically, there should be some insulation constant $k > 0$ so that $u'(t) = k(\sin(t) - u(t))$. I took $k = 1$ for simplicity. We assume no air conditioning.

**5. Hooke's Law.** I claim that the differential equation $x''(t) = -\omega^2 x(t)$ has general solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

where $A$ and $B$ are arbitrary constants.

(a) Verify that this is, indeed, a solution.

(b) Solve for $A$ and $B$ in terms of the initial conditions $x(0)$ and $x'(0)$.

(c) The solution can alternatively be expressed as

$$x(t) = C\cos(\omega(t-\alpha)).$$

Solve for $C$ and $\alpha$ in terms of $x(0)$ and $x'(0)$. [Hint: We can use the same method as in Problem 3. It is based on the angle sum identity:

$$\cos(\omega(t-\alpha)) = \cos(\omega t - \omega\alpha) = \cos(\omega\alpha)\cos(\omega t) + \sin(\omega\alpha)\sin(\omega t).]$$

**6. Euler's Identity.** Let $i$ denote a[2] square root of $-1$. *Euler's identity* provides a connection between exponential and trigonometric functions:

$$\boxed{e^{it} = \cos t + i\sin t.}$$

(a) Use Euler's identity to prove the *angle sum formulas*:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

[Hint: Use the property $e^{i\alpha}e^{i\beta} = e^{i\alpha+i\beta} = e^{i(\alpha+\beta)}$ of exponentials.]

(b) Use Euler's identity to prove that

$$\cos t = \frac{e^{it}+e^{-it}}{2} \quad \text{and} \quad \sin t = \frac{e^{it}-e^{-it}}{2i}.$$

[Hint: First show that $e^{-it} = \cos t - i\sin t$.]

(c) We have seen that the equation $x''(t) = -x(t)$ has general solution

$$x(t) = x(0)\cos t + x'(0)\sin t.$$

I claim that we can also express this solution in the form

$$x(t) = Ae^{it} + Be^{-it}$$

for some constants $A$ and $B$. Use the formulas in part (b) to solve for $A$ and $B$ in terms of $x(0)$ and $x'(0)$. Your answers will involve imaginary numbers.

[2] There are two square roots of $-1$. Pick your favorite and call it $i$. Then the other is called $-i$.