Math 311 — Spring 2023
Practice for Exam 2 — Drew Armstrong

A *linear operator* L takes functions to functions and satisfies

$$L[ay_1(x) + by_2(x)] = aL[y_1(x)] + bL[y_2(x)].$$

In this section we looked at *non-homogeneous linear equations* $L[y(x)] = f(x)$. The general solution is $y(x) = y_c(x) + y_p(x)$ where $y_c(x)$ is the general solution of the homogeneous equation $L[y(x)] = 0$ and $y_p(x)$ is any one particular solution of the non-homogeneous equation.

1. Undetermined Coefficients. In each case find one particular solution:

(a) $x'(t) + x(t) = 5$ [Hint: Guess $x_p(t) = A$.]
(b) $x'(t) + x(t) = \sin(2t)$ [Hint: Guess $x_p(t) = A\cos(2t) + B\sin(2t)$.]
(c) $x'(t) + x(t) = t^2$ [Hint: Guess $x_p(t) = A + Bt + Ct^2$.]
(d) $x''(t) + x(t) = e^{2t}$ [Hint: Guess $x_p(t) = Ae^{2t}$.]

2. Putting It Together. Solve the initial value problems:

(a) $x'(t) + x(t) = 5$; $x(0) = 1$
(b) $x'(t) + x(t) = \sin(2t)$; $x(0) = 1$
(c) $x'(t) + x(t) = t^2$; $x(0) = 1$
(d) $x''(t) + x(t) = e^{2t}$; $x(0) = 1$, $x'(0) = 0$

3. Partial Fractions. Find the partial fraction decomposition of the following expressions:

(a) $\frac{1}{s(s+1)}$
(b) $\frac{1}{s(s-1)(s-2)}$
(c) $\frac{1}{s^2(s^2+1)}$

4. Laplace Transform Rules. Use the information in the table to compute the following Laplace transforms:

(a) $\mathscr{L}[t \cdot e^{2t}]$
(b) $\mathscr{L}[t^2 \cdot e^{2t}]$
(c) $\mathscr{L}[e^{3t} \cdot \cos(2t)]$
(d) $\mathscr{L}[t \cdot \cos t]$
(e) $\mathscr{L}^{-1}\left[\frac{1}{(s-3)^2+4}\right]$
(f) $\mathscr{L}^{-1}[e^{-4s}/s^3]$

5. Using Laplace Transforms. Use Laplace transforms to solve the following initial value problems:

(a) $x'(t) + x(t) = 2$; $x(0) = 0$
(b) $x''(t) - x'(t) = e^{2t}$; $x(0) = x'(0) = 0$
(c) $x'(t) + x(t) = \delta(t-1)$; $x(0) = 1$
(d) $x''(t) + x(t) = t$; $x(0) = x'(0) = 1$
(e) $x''(t) + x(t) = \delta(t-4)$; $x(0) = x'(0) = 0$

6. Eigenvectors. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & -16 \\ 8 & -17 \end{pmatrix}$

7. First Order Linear Systems. Consider the first order linear system:

$$\begin{cases} x'(t) &= 5x(t) - 4y(t), \\ y'(t) &= 6x(t) - 5y(t). \end{cases}$$

(a) Use your answer from 6(a) to find the general solution.
(b) Find the specific solution with $x(0) = 1$ and $y(0) = 2$.

8. Second Order Linear Systems. Consider the second order linear system:

$$\begin{cases} x''(t) &= 7x(t) - 16y(t), \\ y''(t) &= 8x(t) - 17y(t). \end{cases}$$

(a) Use your answer from 6(b) to find the general solution.
(b) Find the particular solution with $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$.

Rules for Laplace Transforms. Let $\mathscr{L}[f(t)] = F(s)$. Then

- $\mathscr{L}[tf(t)] = -F'(s)$
- $\mathscr{L}[e^{at}f(t) = F(s-a)$
- $\mathscr{L}[f'(t)] = sF(s) - f(0)$
- $\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$
- $\mathscr{L}[H(t-a)f(t-a)] = e^{-as}F(s)$, where $H(t-a) = \begin{cases} 0 & t < a, \\ 1 & t > a. \end{cases}$

Table of Laplace Transforms.

- $\mathscr{L}[0] = 0$
- $\mathscr{L}[1] = 1/s$
- $\mathscr{L}[e^{as}] = 1/(s-a)$
- $\mathscr{L}[t] = 1/s^2$
- $\mathscr{L}[t^n] = n!/s^{n+1}$
- $\mathscr{L}[\cos(kt)] = s/(s^2+k^2)$
- $\mathscr{L}[\sin(kt)] = k/(s^2+k^2)$
- $\mathscr{L}[\delta(t)] = 1$, where $\delta(t)$ is the Dirac delta function
- $\mathscr{L}[\delta(t-a)] = e^{-as}$
- $\mathscr{L}[H(t-a)] = e^{-as}/s$

Solutions.

1.

(a) Substituting $x_p(t) = A$ gives $A = 5$.

(b) Substituting $x_p(t) = A\cos(2t) + B\sin(2t)$ gives

$$-2A\sin(2t) + 2B\cos(2t) + A\cos(2t) + B\sin(2t) = \sin(2t).$$

Comparing coefficients gives $A + 2B = 0$ and $B - 2A = 1$, hence $A = -2/5$ and $B = 1/5$.

(c) Substituting $x_p(t) = A + Bt + Ct^2$ gives

$$B + 2Ct + A + Bt + Ct^2 = t^2.$$

Comparing coefficients gives $A = 2$, $B = -2$ and $C = 1$.

(d) Substituting $x_p(t) = Ae^{2t}$ gives

$$4Ae^{2t} + Ae^{2t} = e^{2t}.$$

Comparing coefficients gives $A = 1/5$.

2.

(a) The general homogeneous solution is $x_c(t) = C_1e^{-t}$. The general solution is $x(t) = x_c(t) + x_p(t) = C_1e^{-t} + 5$. The solution with $x(0) = 1$ is

$$x(t) = 5 - 4e^{-t}.$$

(b) The general homogeneous solution is $x_c(t) = C_1e^{-t}$. The general solution is $x(t) = x_c(t) + x_p(t) = C_1e^{-t} + (-2/5)\cos(2t) + (1/5)\sin(2t)$. The solution with $x(0) = 1$ is

$$x(t) = -\frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t) + \frac{7}{5}e^{-t}.$$

(c) The general homogeneous solution is $x_c(t) = C_1e^{-t}$. The general solution is $x(t) = x_c(t) + x_p(t) = C_1e^{-t} + t^2 - 2t + 2$. The solution with $x(0) = 1$ is

$$x(t) = -e^{-t} + t^2 - 2t + 2.$$

(d) The general homogeneous solution is $x_c(t) = C_1\cos t + C_2\sin t$. The general solution is $x(t) = x_c(t) + x_p(t) = C_1\cos t + C_2\sin t + (1/5)e^{2t}$. The solution with $x(0) = 1$ and $x'(0) = 0$ is

$$x(t) = \frac{4}{5}\cos t - \frac{2}{5}\sin t + \frac{1}{5}e^{-t}.$$

3.

(a) We are looking for A, B so that

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}.$$

The solution is $A = 1$ and $B = -1$.

(b) We are looking for A, B, C such that

$$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}.$$

The solution is $A = 1/2$, $B = -1$, $C = 1/2$.

(c) We are looking for A, B, C, D such that

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}.$$

The solution is $A = 0,\ B = 1,\ C = 0,\ D = -1$. Shotrcut: Let $s^2 = r$ and note that

$$\frac{1}{s^2(s^2+1)} = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

4.

(a) $\mathscr{L}[t \cdot e^{2t}] = -\frac{d}{ds}\mathscr{L}[e^{2t}] = -\frac{d}{ds}\frac{1}{s-2} = \frac{1}{(s-2)^2}$

(b) $\mathscr{L}[t \cdot e^{2t}] = \mathscr{L}[t \cdot t \cdot e^{2t}] = -\frac{d}{ds}\mathscr{L}[t \cdot e^{2t}] = -\frac{d}{ds}\frac{1}{(s-2)^2} = \frac{2}{(s-2)^3}$

(c) $\mathscr{L}[e^{3t} \cdot \cos(2t)] = \mathscr{L}[\cos(2t)]_{s \to s-3} = \left[\frac{s}{s^2+4}\right]_{s \to s-3} = \frac{s-3}{(s-3)^2+4}$

(d) $\mathscr{L}[t \cdot \cos t] = -\frac{d}{ds}\mathscr{L}[\cos t] = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$

(e) We know that $\mathscr{L}[\sin(2t)] = \frac{2}{s^2+4}$, hence $\mathscr{L}[e^{3t} \cdot \sin(2t)] = \frac{2}{(s-3)^2+4}$, hence

$$\mathscr{L}^{-1}\left[\frac{1}{(s-3)^2+4}\right] = \frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{(s-3)^2+4}\right] = \frac{1}{2} \cdot e^{3t} \cdot \sin(2t).$$

(f) If $\mathscr{L}[f(t)] = F(s)$ then we know that $\mathscr{L}^{-1}[e^{-as} \cdot F(s)] = H(t-a)f(t-a)$. In our case we let $F(s) = 1/s^3$, so that $f(t) = \mathscr{L}^{-1}[1/s^3] = \frac{1}{2}\mathscr{L}^{-1}[2/s^3] = \frac{1}{2}t^2$. Hence

$$\mathscr{L}^{-1}\left[e^{-4s} \cdot \frac{1}{s^3}\right] = \mathscr{L}^{-1}\left[e^{-4s} \cdot F(s)\right] = H(t-4)f(t-4) = H(t-4)\frac{1}{2}(t-4)^2.$$

5.

(a) Applying Laplace transforms gives

$$\begin{aligned} sX - 0 + X &= 2/s \\ X &= \frac{2}{s(s+1)} \\ X &= 2\frac{1}{s} - 2\frac{1}{s+1} \\ x(t) &= 2 - 2e^{-t}. \end{aligned}$$

(b) Applying Laplace transforms gives

$$\begin{aligned} s^2X - 0s - 0 - (sX - 0) &= \frac{1}{s-2} \\ X &= \frac{1}{s(s-1)(s-2)} \\ X &= \frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2} \\ x(t) &= \frac{1}{2} - e^t + \frac{1}{2}e^{2t}. \end{aligned}$$

(c) Applying Laplace transforms gives

$$\begin{aligned} sX - 1 + X &= e^{-s} \\ X &= \frac{1}{s+1} + e^{-1s} \cdot \frac{1}{s+1} \end{aligned}$$

$$x(t) = e^{-t} + \mathscr{L}^{-1}\left[e^{-1s}\cdot\frac{1}{s+1}\right]$$

Take $F(s) = \frac{1}{s+1}$ so that $f(t) = \mathscr{L}^{-1}[F(s)] = e^{-t}$. Then

$$\begin{aligned} x(t) &= e^{-t} + \mathscr{L}^{-1}\left[e^{-1s}\cdot\frac{1}{s+1}\right] \\ &= e^{-t} + \mathscr{L}^{-1}[e^{-1s}F(s)] \\ &= e^{-t} + H(t-1)f(t-1) \\ &= e^{-t} + H(t-1)e^{-(t-1)}. \end{aligned}$$

(d) Applying Laplace transforms gives

$$\begin{aligned} s^2X - s - 1 + X &= \frac{1}{s^2} \\ X &= \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)} \\ X &= \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1} \\ X &= \frac{s}{s^2+1} + \frac{1}{s^2} \\ x(t) &= \cos t + t. \end{aligned}$$

(e) Applying Laplace transforms gives

$$\begin{aligned} s^2X - 0s - 0 + X &= e^{-4s} \\ X &= e^{-4s}\frac{1}{s^2+1} \\ x(t) &= \mathscr{L}^{-1}\left[e^{-4s}\cdot\frac{1}{s^2+1}\right]. \end{aligned}$$

Let $F(s) = 1/(s^2+1)$ so that $f(t) = \mathscr{L}^{-1}[F(s)] = \sin t$. Then

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[e^{-4s}\cdot\frac{1}{s^2+1}\right] \\ &= \mathscr{L}^{-1}\left[e^{-4s}\cdot F(s)\right] \\ &= H(t-4)f(t-4) \\ &= H(t-4)\sin(t-4). \end{aligned}$$

6.

(a) We have

$$\begin{pmatrix}5 & -4\\ 6 & -5\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 1\begin{pmatrix}1\\1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}5 & -4\\ 6 & -5\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = -1\begin{pmatrix}2\\3\end{pmatrix}$$

(b) We have

$$\begin{pmatrix}7 & -16\\ 8 & -17\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = -1\begin{pmatrix}2\\1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}7 & -16\\ 8 & -17\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = -9\begin{pmatrix}1\\1\end{pmatrix}.$$

7.

(a) The general solution of the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t}.$$

(b) Substituting $x(0) = 1$ and $y(0) = 2$ gives $c_1 = -1$ and $c_2 = 1$.

8.

(a) The frequencies corresponding to $\lambda_1, \lambda_2 = -1, -9$ are $\omega_1, \omega_2 = 1, 3$. The general solution of the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t + b_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + a_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(3t) + b_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(3t).$$

(b) Substituting $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$ gives $a_1 = a_2 = 0$, $b_1 = -1$ and $b_2 = 2/3$, hence

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \frac{2}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(3t).$$