Math 311 **Exam 1**
Spring 2023 **Fri Mar 3**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

Problem 1. Let $h(t)$ be the height of a projectile above the surface of a planet. Assume that $h(t)$ satisfies the differential equation $h''(t) = -2$. Suppose the projectile is launched upwards with initial height $h(0) = 5$ and initial velocity $h'(0) = 4$.

(a) Solve for $h(t)$.

We integrate $h''(t)$ twice to get $h(t)$:

$$\begin{aligned} h''(t) &= -2 \\ h'(t) &= \int -2\,dt \\ &= -2t + c_1, \\ h(t) &= \int (-2t + c_1)\,dt \\ &= -t^2 + c_1 t + c_2. \end{aligned}$$

Substituting $t = 0$ into $h'(t)$ gives

$$4 = h'(0) = -2(0) + c_1 = c_1$$

and substituting $t = 0$ into $h(t)$ gives

$$5 = h(0) = -(0)^2 + c_1(0) + c_2 = c_2.$$

Hence the solution is

$$\boxed{h(t) = -t^2 + 4t + 5.}$$

(b) When does the projectile hit the ground?

The projectile hits the ground when $h(t) = 0$. From part (a) this means

$$\begin{aligned} -t^2 + 4t + 5 &= 0 \\ t^2 - 4t - 5 &= 0 \\ (t-5)(t+1) &= 0. \end{aligned}$$

So the projectile hits the ground when $t = 5$ or $t = -1$. (The solution $t = -1$ is not physically relevant.)

Problem 2. Consider the differential equation $dy/dx = 2 - y$.

(a) Find a formula for the solution with $y(0) = 1$.

We use separation of variables:

$$\begin{aligned} dy/dx &= 2 - y \\ \frac{dy}{2-y} &= dx \\ \int \frac{dy}{2-y} &= \int dx + C \\ -\ln(2-y) &= x + C && \text{substitute } u = 2 - y \\ \ln(2-y) &= -x - C \\ 2 - y &= e^{-x-C} \\ y &= 2 - e^{-x-C} \\ y &= 2 + Ae^{-x} \end{aligned}$$

for some constant A. Substituting $t = 0$ gives

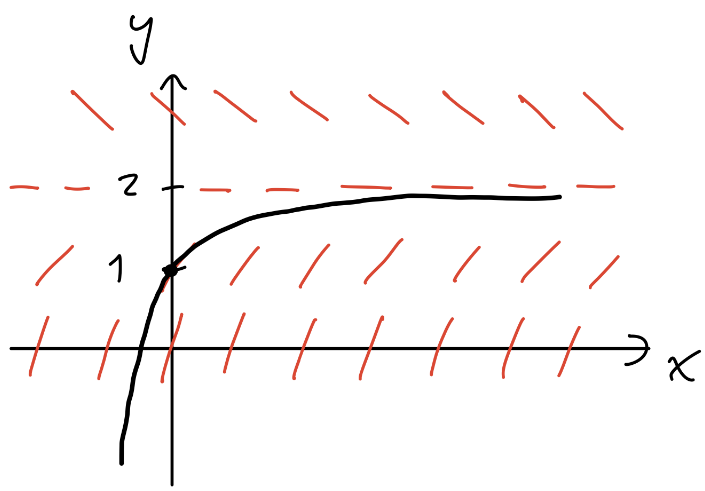
$$1 = y(0) = 2 + Ae^0 = 2 + A \quad \Longrightarrow \quad A = -1.$$

So the solution is

$$\boxed{y(x) = 2 - e^{-x}.}$$

(b) Sketch the slope field of the equation and your solution from part (a).

At each point (x, y) we draw a little line of slope $2 - y$:



Problem 3. Consider the differential equation $dy/dx = y^2$.

(a) Find a formula for the solution with $y(0) = 1$.

We use separation of variables:

$$\begin{aligned} dy/dx &= y^2 \\ \frac{dy}{y^2} &= dx \\ \int \frac{dy}{y^2} &= \int dx + C \\ -\frac{1}{y} &= x + C \\ y &= -\frac{1}{x+C} \end{aligned}$$

for some constant C. Substituting $t = 0$ gives

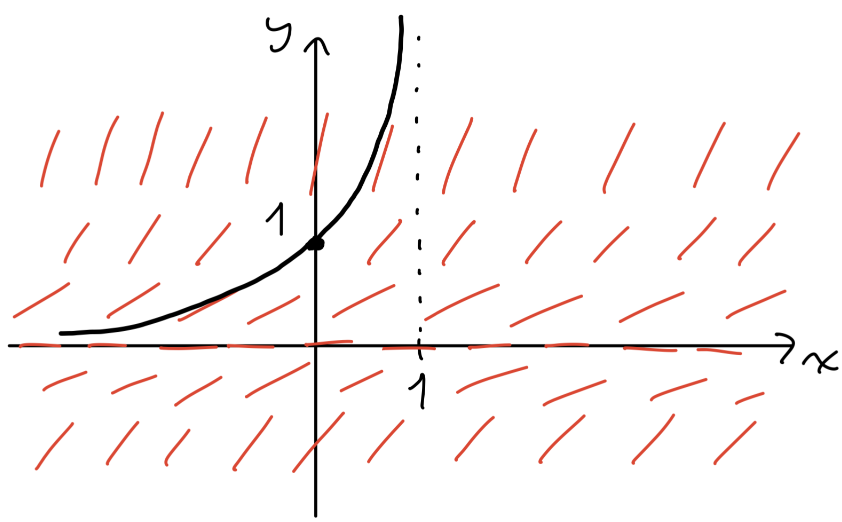
$$1 = y(0) = -1/(0 + C) = -1/C \quad \Longrightarrow \quad C = -1.$$

So the solution is

$$\boxed{y(x) = -\frac{1}{x-1} = \frac{1}{1-x}.}$$

(b) Sketch the slope field of the equation and your solution from part (a).

At each point (x, y) we draw a little line of slope y^2:



Note that the solution with $y(0) = 1$ has a vertical asymptote at $x = 1$. We know this from the formula in part (a). It would be difficult to see this just by looking at the slope field.

Problem 4. Consider the differential equation $dy/dx = x + y$.

(a) Find a formula for the solution with $y(0) = 1$.

We use the method of integrating factors. First we write the equation as

$$\frac{dy}{dx} - 1y = x$$
$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x) = -1$ and $Q(x) = x$. The integrating factor is

$$\rho(x) = \exp\left(\int P(x)\,dx\right) = \exp\left(\int -1\,dx\right) = \exp(-x) = e^{-x}.$$

Multiply both sides by $\rho(x) = e^{-x}$ and observe that the left side simplifies:

$$\begin{aligned} y' - y &= x \\ e^{-x}(y' - y) &= xe^{-x} \\ e^{-x}y' - e^{-x}y &= xe^{-x} \\ (e^{-x}y)' &= xe^{-x}. \end{aligned}$$

Now integrate both sides:

$$\begin{aligned} e^{-x}y &= \int xe^{-x}\,dx + C \\ e^{-x}y &= -(x+1)e^{-x} + C && \text{integration by parts} \\ y &= \frac{-(x+1)e^{-x} + C}{e^{-x}} \\ y &= -x - 1 + Ce^{x} \end{aligned}$$

for some constant C. Substituting $y(0) = 1$ gives

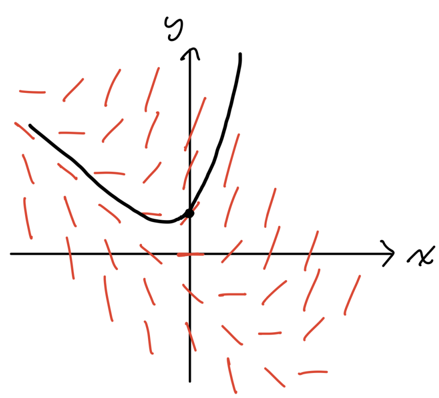
$$1 = y(0) = -0 - 1 + Ce^0 = -1 + C \implies C = 2.$$

So the solution is

$$\boxed{y(x) = -x - 1 + 2e^{x}.}$$

(b) Sketch the slope field of the equation and your solution from part (a).

At every point (x, y) we draw a little line of slope $x + y$:



Problem 5. Consider the differential equation $x''(t) - 2x'(t) + 2x(t) = 0$.

(a) Find a formula for the general solution.

First we look for solutions of the form $x(t) = e^{\lambda t}$:

$$\begin{aligned} x''(t) - 2x'(t) + 2x(t) &= 0 \\ \lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} + 2e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 - 2\lambda + 2) &= 0 \\ \lambda^2 - 2\lambda + 2 &= 0. \end{aligned}$$

The solution of this "characteristic equation" is

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Hence the general solution of the differential equation is

$$\begin{aligned} x(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ &= c_1 e^{(1+i)t} + c_2 e^{(1-i)t} \\ &= c_1 e^t e^{it} + c_2 e^t e^{-it} \\ &= e^t (c_1 e^{it} + c_2 e^{-it}) \end{aligned}$$

for some constants c_1 and c_2. After applying Euler's formula we can write this as

$$x(t) = e^t\,(A\cos t + B\sin t)$$

for some constants A and B.

(b) Find the specific solution with $x(0) = 1$ and $x'(0) = 2$.

To solve for A and B we first substitute $t = 0$ into $x(t)$:

$$1 = x(0) = e^0\,(A\cos 0 + B\sin 0) = A.$$

Then we compute $x'(t)$ using the product rule and substitute $t = 0$:

$$\begin{aligned} x(t) &= e^t(\cos t + B\sin t) \\ x'(t) &= e^t(\cos t + B\sin t) + e^t(-\sin t + B\cos t) \\ x'(0) &= e^0(\cos 0 + B\sin 0) + e^0(-\sin 0 + B\cos 0) \\ 2 &= 1(1+0) + 1(0+B) \\ 2 &= 1 + B \\ B &= 1. \end{aligned}$$

So the solution is

$$\boxed{x(t) = e^t\,(\cos t + \sin t)\,.}$$