Math 311 **Spring 2023**
Practice for Exam 1 **Drew Armstrong**

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1. Slope Fields. Just sketch. Don't solve.

(a) Sketch the slope field of $dy/dx = y$ and the two solutions with $y(0) = 1, -1$.
(b) Sketch the slope field of $dy/dx = x$ and the two solutions with $y(0) = 1, -1$.
(c) Sketch the slope field of $dy/dx = 5 - y$ and the two solutions with $y(0) = 4, 6$.
(d) Sketch the slope field of $dy/dx = y(5 - y)$ and the three solutions with $y(0) = -1, 1, 6$.

2. Direct Integration. Solve the initial value problems.

(a) $x'' = 5$; $x(0) = 1$, $x'(0) = 3$
(b) $x'' = t$; $x(0) = 1$, $x'(0) = 3$
(c) $x' = \sin t$; $x(0) = 1$
(d) $x' = e^{3t}$; $x(0) = 4$

3. Free Fall (No Air Resistance). Suppose that the height of a projectile near the surface of a planet satisfies $h''(t) = -1$. Suppose the projectile is launched from a height $h(0) = 1$ with initial speed $h'(0) = 4$ (up). When does the projectile hit the ground ($h = 0$)?

4. Separation of Variables. Solve the initial value problems.

(a) $dy/dx = 2y - 3$; $y(0) = 1$
(b) $dy/dx = y^2$; $y(0) = 1$
(c) $dy/dx = xy$; $y(0) = 1$
(d) $dy/dx = x/y$; $y(0) = 1$
(e) $dy/dx = y(1 - y)$; $y(0) = 1/2$. [Hint: $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$.]

5. Newton's Law of Cooling. Suppose the temperature of a cup of coffee satisfies $u'(t) = 5 - u(t)$ and $u(0) = 7$. Find a formula for $u(t)$ and compute the limit of $u(t)$ as $t \to +\infty$.

6. Free Fall (Air Resistance). Suppose the velocity of a projectile near the surface of a planet satisfies $dv/dt = -1 - v$. Suppose the projectile is dropped, so that $v(0) = 0$. Find a formula for $v(t)$ and compute the limit of $v(t)$ as $t \to +\infty$.

7. Integration Factors. Solve the initial value problems.

(a) $x' + x = 1$; $x(0) = 4$
(b) $x' + x = t$; $x(0) = 4$
(c) $x' = tx + t$; $x(0) = 4$

8. Trigonometry.

(a) Express $3\cos t + 4\sin t$ in the form $C\cos(t - \alpha)$.
(b) Express $(2 + i)e^{it} + (2 - i)e^{-it}$ in the form $A\cos t + B\sin t$.

9. Linear, Homogeneous, Constant Coefficients. Solve the initial value problems.

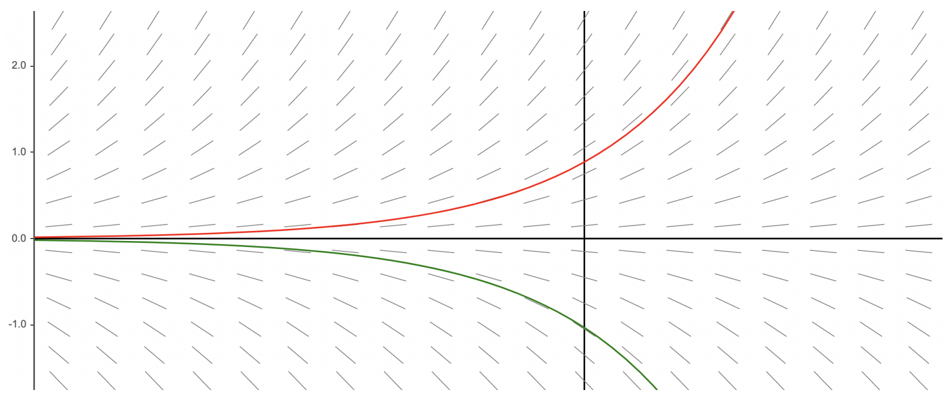
(a) $y'' - y = 0$; $y(0) = 1$; $y'(0) = 2$
(b) $y'' + 4y' + 4y = 0$; $y(0) = 1$; $y'(0) = 2$
(c) $y'' + 4y' + 5y = 0$; $y(0) = 1$; $y'(0) = 2$

10. Damped Oscillations. The equation $mx'' + \gamma x' + kx = 0$ represents a damped oscillator with mass $m > 0$, stiffness $k > 0$ and friction $\gamma > 0$. Assume that $\gamma^2 - 4mk < 0$ so the characteristic equation $m\lambda^2 + \gamma\lambda + k = 0$ has complex roots. In this case make a rough sketch of the solution with $x(0) = 1$ and $x'(0) = 1$. (You do not need to compute a formula.)

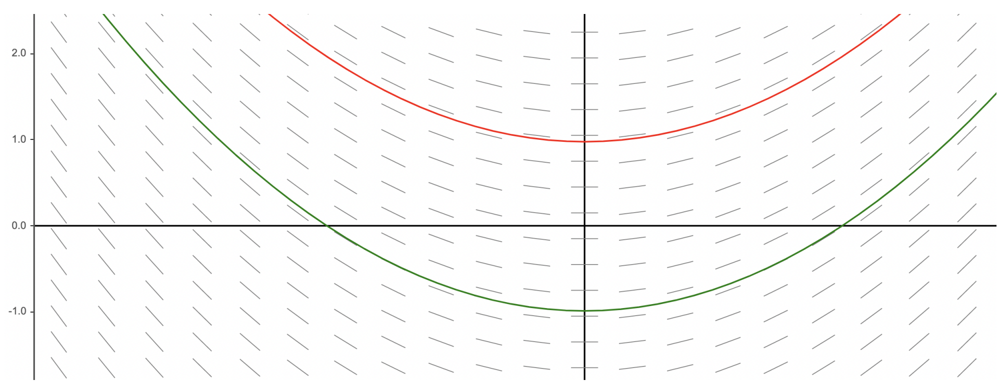
Solutions.

1. Slope Fields.

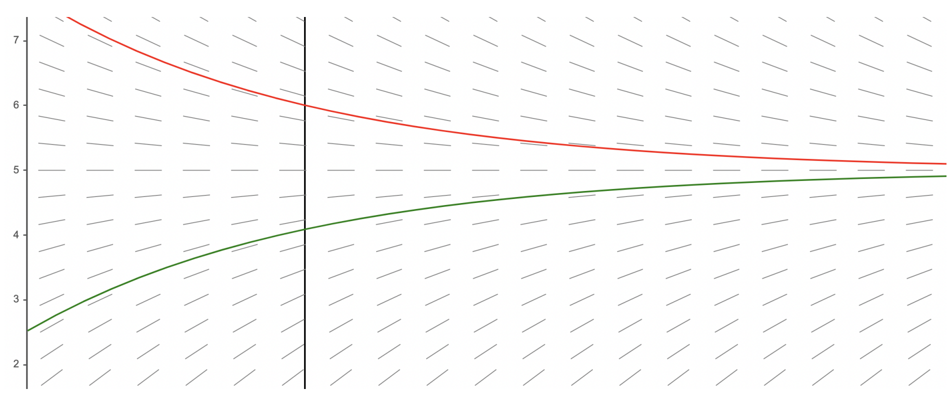
(a)



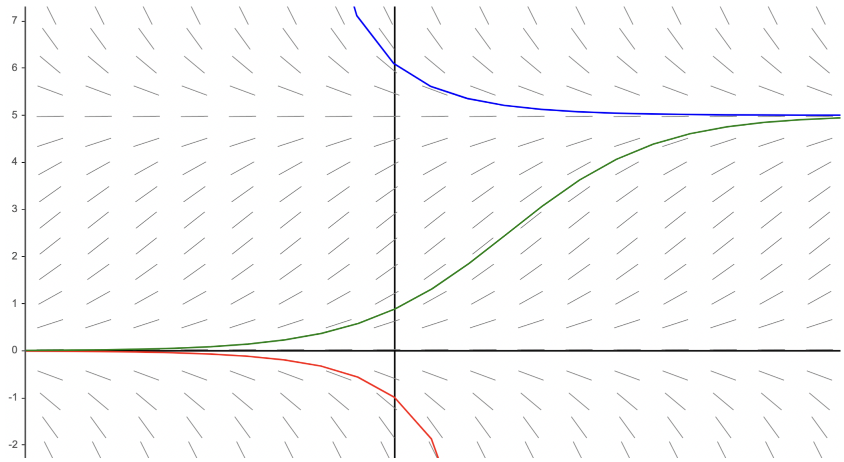
(b)



(c)



(d)



2. Direct Integration.

(a) $x(t) = (5/2)t^2 + 3t + 1$
(b) $x(t) = (1/6)t^3 + 3t + 1$
(c) $x(t) = 2 - \cos t$
(d) $x(t) = e^{3t}/3 + 11/3$

3. Free Fall (No Air Resistance). The solution of $h''(t) = -1$ with $h(0) = 1$ and $h'(0) = 4$ is

$$h(t) = -\frac{1}{2}t^2 + 4t + 1.$$

The solution of $h(t) = 0$ is $t = 4 \pm 3\sqrt{2}$. Only positive time is physically relevant, so the solution is $t = 4 + 3\sqrt{2}$.

4. Separation of Variables.

(a) $y(x) = 3/2 - e^{2x}/2$
(b) $y(x) = 1/(1 - x)$
(c) $y(x) = e^{x^2/2}$
(d) $y(x) = \sqrt{x^2 + 1}$
(e) $y(x) = 1/(1 + e^{-x})$

5. Newton's Law of Cooling. The solution of $u'(t) = 5 - u(t)$ with $u(0) = 7$ is

$$u(t) = 5 + 2e^{-t}.$$

Since $e^{-t} \to 0$ we note that $u(t) \to 5$ as $t \to \infty$. (The temperature of the coffee approaches the room temperature 5.)

6. Free Fall (Air Resistance). The solution of $dv/dt = -1 - v$ with $v(0) = 0$ is

$$v(t) = -1 + e^{-t}.$$

Since $e^{-t} \to 0$ we note that $v(t) \to -1$ as $t \to \infty$. (The velocity approaches the terminal velocity -1.)

7. Integration Factors.

(a) $x(t) = 1 + 3e^{-t}$
(b) $x(t) = t - 1 + 5e^{-t}$

(c) $x(t) = -1 + 5e^{t^2/2}$

8. Trigonometry.

(a) The general formulas for $A\cos t + B\sin t = C\cos(t-\alpha)$ are $A = C\cos\alpha$ and $B = C\sin\alpha$, which imply that $C = \sqrt{A^2+B^2}$ and $\alpha = \tan^{-1}(B/A)$. In our case we have $A = 3$ and $B = 4$, so that

$$3\cos t + 4\sin t = 5\cos\left(t - \tan^{-1}(4/3)\right).$$

(b) Recall Euler's formulas:

$$\begin{aligned} e^{it} &= \cos t + i\sin t \\ e^{-it} &= \cos t - i\sin t. \end{aligned}$$

Hence

$$\begin{aligned} &(2+i)e^{it} + (2-i)e^{-it} \\ &\quad = (2+i)(\cos t + i\sin t) + (2-i)(\cos t - i\sin t) \\ &\quad = 2(\cos t + i\sin t) \\ &\qquad + i(\cos t + i\sin t) \\ &\qquad + 2(\cos t - i\sin t) \\ &\qquad - i(\cos t - i\sin t) \\ &\quad = 2\cos t + \cancel{2i\sin t} \\ &\qquad + \cancel{i\cos t} - \sin t \\ &\qquad + 2\cos t - \cancel{2i\sin t} \\ &\qquad - \cancel{i\cos t} - \sin t \\ &\quad = 4\cos t - 2\sin t. \end{aligned}$$

9. Linear, Homogeneous, Constant Coefficients.

(a) The characteristic equation of $y'' - y = 0$ is $\lambda^2 - 1 = 0$ which has solutions $\lambda = \pm 1$. So the general solution is

$$y(x) = Ae^{t} + Be^{-t}.$$

Substituting $y(0) = 1$ into $y(x)$ gives $1 = A + B$ and substituting $y'(0) = 2$ into $y'(t) = Ae^{t} - Be^{-t}$ gives $2 = A - B$. Putting these together gives $A = 3/2$ and $B = -1/2$, hence

$$\boxed{y(x) = \frac{3}{2}e^{t} - \frac{1}{2}e^{-t}.}$$

(b) The characteristic equation of $y'' + 4y' + 4y = 0$ is $\lambda^2 + 4\lambda + 4 = 0$, which factors as $(\lambda+2)^2 = 0$. Since $\lambda = -2$ is a repeated root, the general solution is

$$y(x) = Ae^{-2x} + Bxe^{-2x}.$$

Substituting $y(0) = 1$ into $y(x)$ gives $1 = A$ and substituting $y'(0) = 2$ into

$$y'(x) = -2e^{-2x} + Be^{-2x} + Bx(-2e^{-2x})$$

gives $2 = -2 + B$, hence $B = 4$. The solution is

$$\boxed{y(x) = e^{-2x} + 4xe^{-2x}.}$$

(c) The characteristic equation of $y'' + 4y' + 5y = 0$ is $\lambda^2 + 4\lambda + 5 = 0$. The quadratic formula gives $\lambda = -2 \pm i$. The general solution in complex form is

$$y(x) = c_1 e^{(-2+i)t} + c_2 e^{(-2-i)t},$$

which can also be expressed in real form as

$$y(x) = e^{-2t} \left(A \cos t + B \sin t \right).$$

Substituting $y(0) = 1$ into $y(x)$ gives $1 = A$ and substituting $y'(0) = 2$ into

$$y'(x) = -2e^{-2x} \left(1 \cos t + B \sin t \right) + e^{-2x} \left(-1 \sin t + B \cos t \right)$$

gives $2 = -2 + B$, hence $B = 4$. The solution is

$$\boxed{y(x) = e^{-2t} \left(\cos t + 4 \sin t \right).}$$

10. Damped Oscillations. Complex roots mean oscillation. The exact formulas for the amplitude and phase are complicated, but it's easy to make a rough sketch: