HW 2 Comments :

Problem 3: If you have

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$
$$\vec{v}(0) = \langle u_0, v_0 \rangle$$
$$\vec{a}(t) = \langle 0, -g \rangle$$

Then,

$$\vec{r}(t) = \left\langle x_0 + u_0 t,\ y_0 + v_0 t - \tfrac{1}{2} g t^2 \right\rangle$$
$$= \left\langle 0 + t\cos\theta,\ 0 + t\sin\theta - \tfrac{1}{2} g t^2 \right\rangle$$

HW 2 due tomorrow (Fri)

No class Mon.

Quiz 2 on Tues beginning of class.

Start Chapter 4 early.

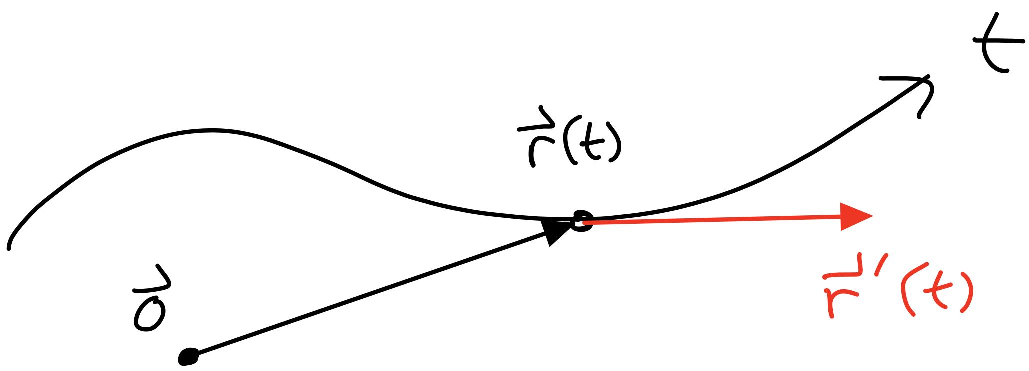
Chap 4: Differentiation of functions of several values.

(i.e. "gradient vectors").

So far we have studied functions

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

Picture: Parametrized path



The derivative is velocity:

$$\vec{r}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle$$

$$\vec{r}'(t) = \langle x_1'(t), x_2'(t), \ldots, x_n'(t) \rangle$$

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt} \right\rangle.$$

Applications:

- Integrate $\|\vec{r}'(t)\|$ to get arc length.
- Newton's 2nd Law.

$$\vec{v}(t) = \vec{r}'(t)$$
$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$
$$\vec{F}(t) = m\,\vec{a}(t) = m\,\vec{r}''(t).$$

Now we consider functions

$$f : \mathbb{R}^n \to \mathbb{R}.$$

To each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ we assign a number (i.e. scalar)

$$f(x_1, x_2, \ldots, x_n).$$

Called a "scalar field".

e.g. temperature
pressure
density
etc.

Example: $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \left(\frac{x}{2}\right)^2 + y^2$$

We can visualize this in 2 ways:

- Level Curves:

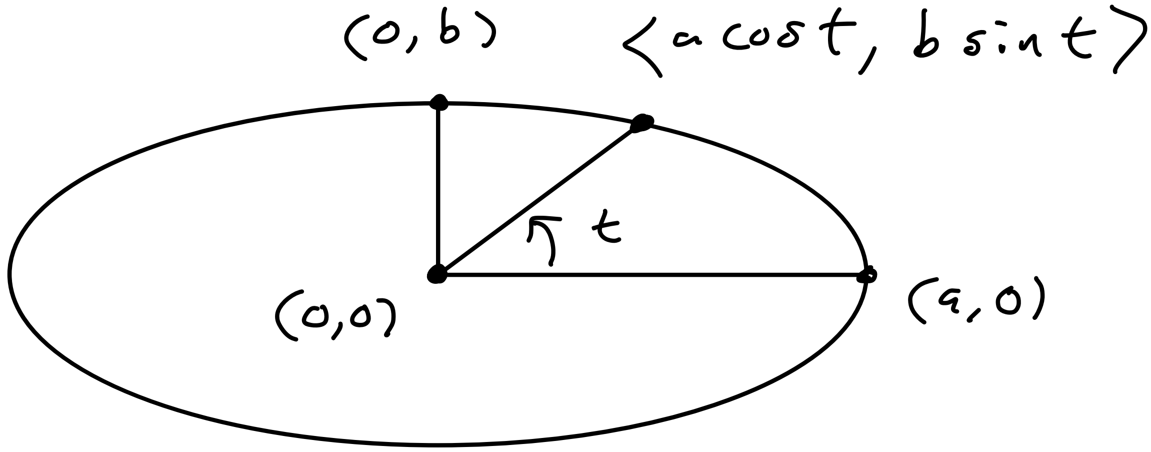
For each fixed constant $c \in \mathbb{R}$ we get a curve in $\mathbb{R}^2$ defined by

$$f(x,y) = c.$$

In our case, each level curve is an ellipse

$$\left(\frac{x}{2}\right)^2 + y^2 = c$$

[Aside: Equation of an ellipse.



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Special case: $a = b = r$

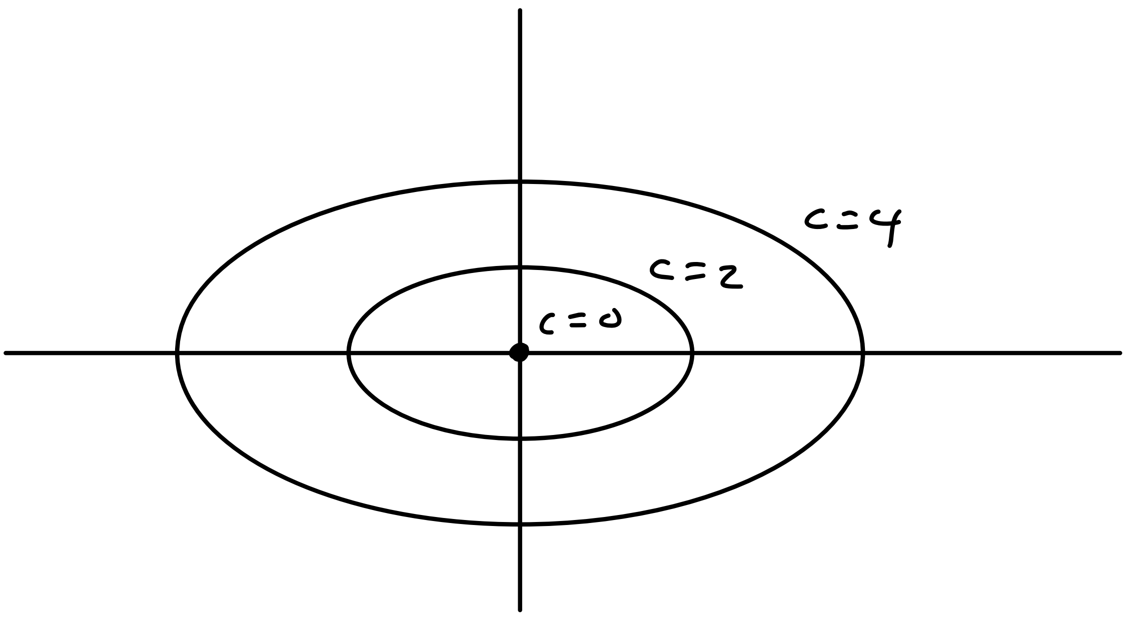
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$x^2 + y^2 = r^2$$

circle of radius r.]

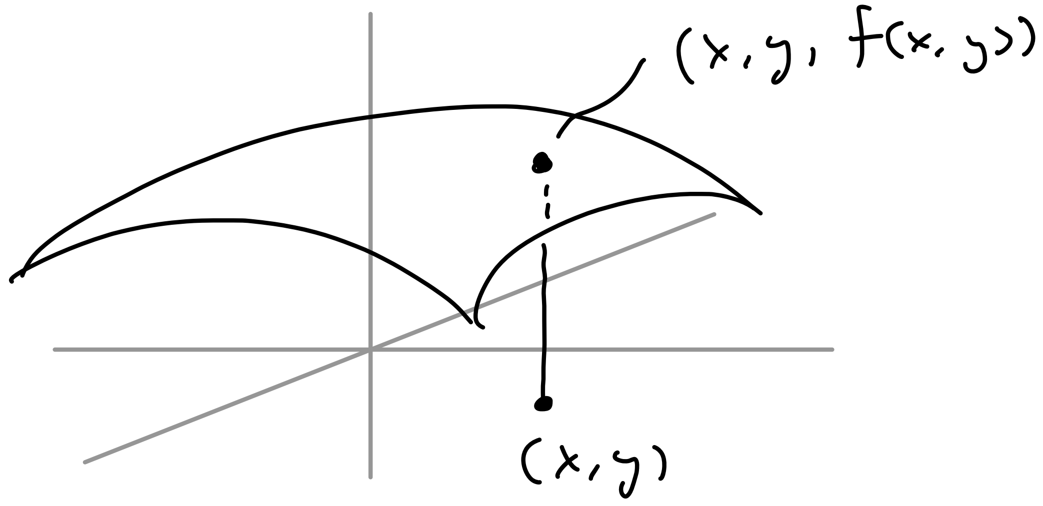
So $\left(\frac{x}{2}\right)^2 + y^2 = c$

$$\left(\frac{x}{2\sqrt{c}}\right)^2 + \left(\frac{y}{\sqrt{c}}\right)^2 = 1.$$



Infinitely many level curves. You have to imagine (or use color, heat map if c = temperature).

• We can also visualize $f(x,y)$ as the 2D surface $z = f(x,y)$.

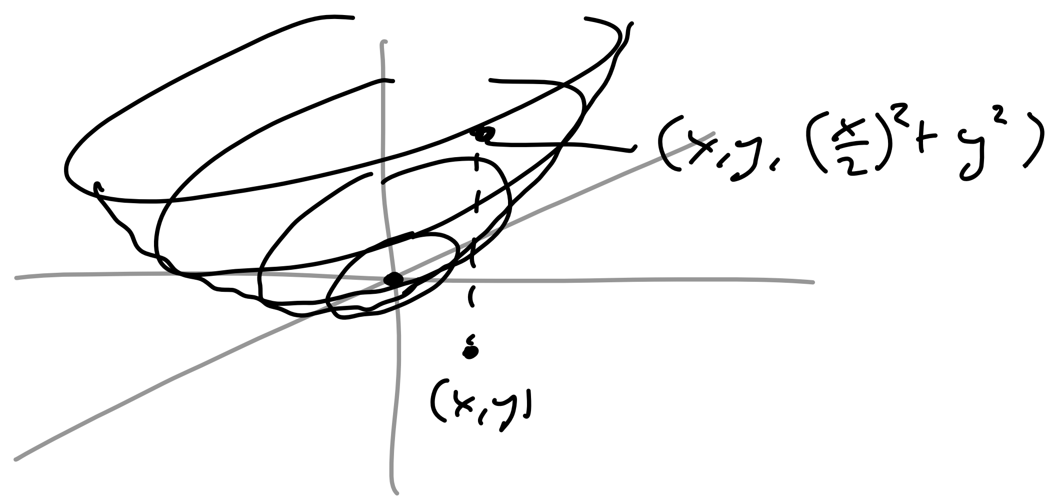


The height of the surface above the point (x,y) is the scalar $f(x,y)$.

[Think: Height = temperature ...]

Our Example: $z = \left(\frac{x}{2}\right)^2 + y^2$

is a parabolic bowl ("paraboloid")



Remark: The graph $z = f(x,y)$ is a useful visualization but it does not work for scaler fields $f: \mathbb{R}^3 \to \mathbb{R}$.

Say $f(x,y,z)$ = temperature at point (x,y,z) in $\mathbb{R}^3$.

Then the "graph" $w = f(x,y,z)$ is a "curvy 3D shape" living in 4D space, which is NOT HELPFUL. In this case our only hope is to visualize the "level surfaces"

$$f(x,y,z) = c \quad \text{for fixed } c.$$

These are the points in $\mathbb{R}^3$ with a given fixed temperature c.

BIG QUESTION:

What is the "derivative" of a scalar field $f: \mathbb{R}^n \to \mathbb{R}$?

Ideas:

(1) If f is temperature at a point then derivative should be "rate of change of temperature".

Problem: Rate of change of temp depends on our <u>velocity</u>.

(1)' Given a point P and a velocity $\vec{v}$, what rate of change of temperature do you feel?

(2) Recall: df/dx is the slope of the tangent line to the curve $y = f(x)$ in the plane.

So... deriv of $f(x,y)$ should be "slope of tangent plane" to the surface $z = f(x,y)$.

Problem: A plane is not determined by a slope; it is determined by a normal vector.

② ′ Derivative of $f(x,y)$ should give the normal vector to the tangent plane to surface $z = f(x,y)$ at a given point.

Good News: Turns out problems ①′ & ②′ are the same, determined by the GRADIENT VECTOR.

Def: Given $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x_1, x_2, \dots, x_n)$$

we define the gradient vector

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \dots, \frac{df}{dx_n} \right\rangle$$

Example: $f(x,y) = \frac{1}{4}x^2 + y^2$

$$\nabla f(x,y) = \left\langle \frac{1}{2}x, 2y \right\rangle$$

[$\frac{d}{dx}\left(\frac{1}{4}x^2 + y^2\right) = \frac{1}{4}2x + 0$

Because d/dx treats y as a constant. Most books write $\frac{\partial}{\partial x}$ instead of $\frac{d}{dx}$ and call this a "partial derivative with respect to x".]

Remarks:

- ∇ is called "nabla".
- The vector $\nabla f(x,y)$ changes from point to point, so we can

think of ∇f as a "vector field"

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

Called the "gradient vector field".

[Examples of vector fields.

force fields (gravity, electric, ...)

velocity fields (wind)]

The gradient vector to

$$f(x,y) = \frac{1}{4}x^2 + y^2$$

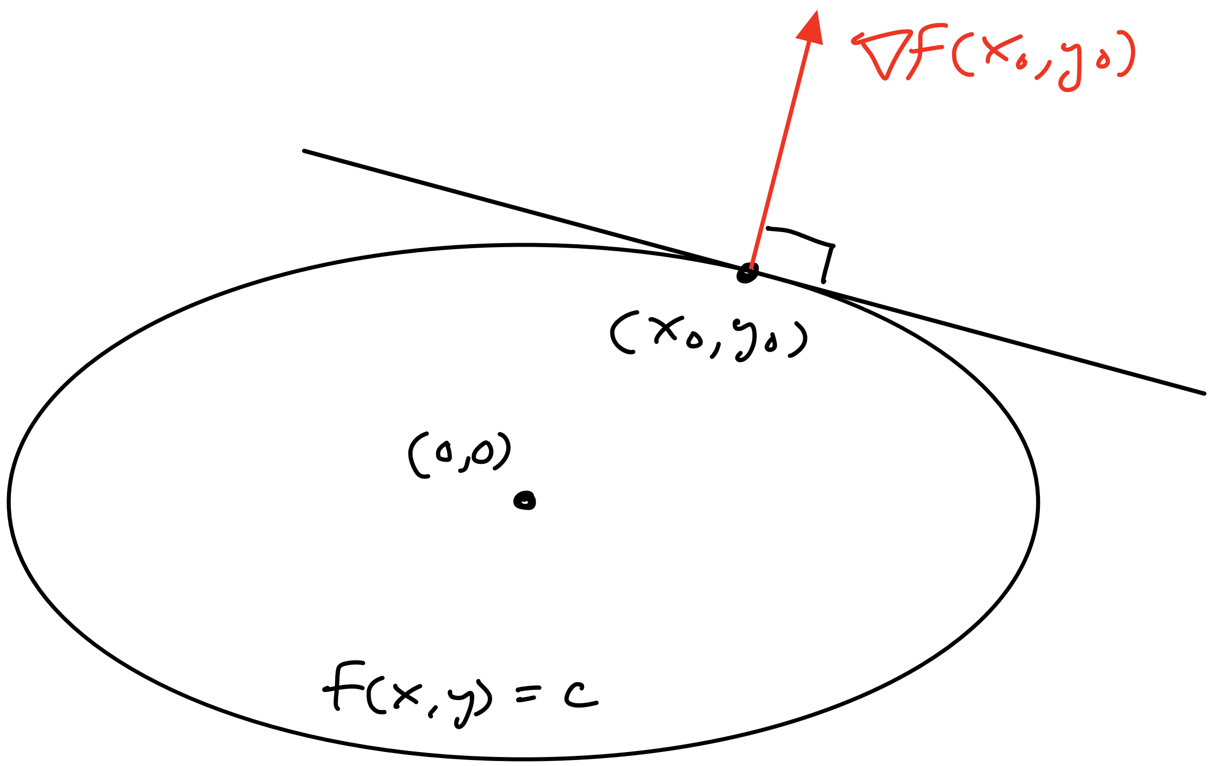
at the point (x_0, y_0) is

$$\nabla f(x_0, y_0) = \left\langle \frac{1}{2}x_0, 2y_0 \right\rangle.$$

MEANING: If $c = f(x_0, y_0)$

is the temperature at this point

then $\nabla f(x_0, y_0)$ is perpendicular to the level curve:



Meaning: $\nabla f(x_0, y_0)$ is the direction of maximum increase of temperature at the point (x_0, y_0).

Two Rigorous Statements.

- The tangent line to the curve $f(x,y) = \text{constant} = c$

at the point (x_0, y_0) is

$$\frac{dF}{dx}(x_0,y_0)(x-x_0) + \frac{dF}{dy}(x_0,y_0)(y-y_0) = 0.$$

$$\nabla f(x_0,y_0) \cdot (x-x_0, y-y_0) = 0$$

this is a normal vector

this is a point on the line.

In our case

$$\nabla f(x_0,y_0) \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$\langle \tfrac{1}{2}x_0, 2y_0 \rangle \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$\frac{1}{2}x_0(x-x_0) + 2y_0(y-y_0) = 0$$

The tangent line to the curve

$$\frac{1}{4}x^2 + y^2 = \text{constant}$$

at the point (x_0, y_0).

e.g. The tangent line to ellipse

$$\frac{1}{4}x^2 + y^2 = 1$$

at point $(\underset{x_0}{\sqrt{2}}, \underset{y_0}{\sqrt{2}/2})$ has equation

$$\frac{1}{2}x_0(x-x_0) + 2y_0(y-y_0) = 0$$

$$\frac{1}{2}\sqrt{2}(x-\sqrt{2}) + 2\frac{\sqrt{2}}{2}\left(y-\frac{\sqrt{2}}{2}\right) = 0$$

$$\frac{\sqrt{2}}{2}x - 1 + \sqrt{2}y - 1 = 0$$

$$\frac{\sqrt{2}}{2}x + \sqrt{2}y = 2$$

$$\frac{1}{2}x + y = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Example: Find equation of the tangent line to circle

$$x^2 + y^2 = 25$$

at the point $(x,y) = (3,4)$.

Let $f(x,y) = x^2 + y^2$.

Then $\nabla f(x,y) = \langle 2x, 2y \rangle$.

$\nabla f(3,4) = \langle 6, 8 \rangle$.

So tangent line is

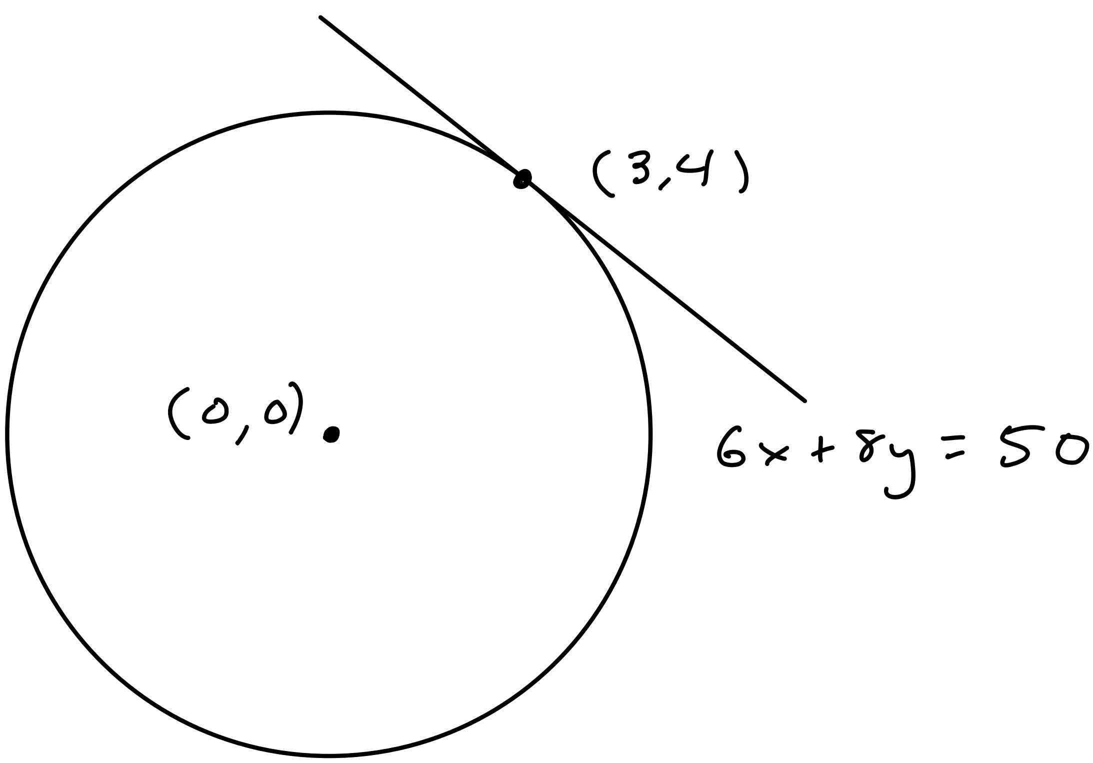
$$\nabla f(3,4) \cdot \langle x-3, y-4 \rangle = 0$$

$$\langle 6, 8 \rangle \cdot \langle x-3, y-4 \rangle = 0$$

$$6(x-3) + 8(y-4) = 0.$$

$$6x + 8y - 18 - 32 = 0$$

$$6x + 8y = 50$$



This Week: Gradients.

(Chapter 4: Differentiation of multivariable functions)

A function $f: \mathbb{R}^n \to \mathbb{R}$ is called a "scalar field". To each point $P = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ it associates a scalar

$$f(P) \text{ or } f(x_1, x_2, \ldots, x_n).$$

Think: $f(P)$ is temperature at the point P.

The derivative of f defined as

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \ldots, \frac{df}{dx_n} \right\rangle$$

$$\left(\text{or } \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \right)$$

Notation: $\frac{\partial f}{\partial x_i}$ means we take deriv. with respect to variable x_i, pretend that the other vars. are constant.

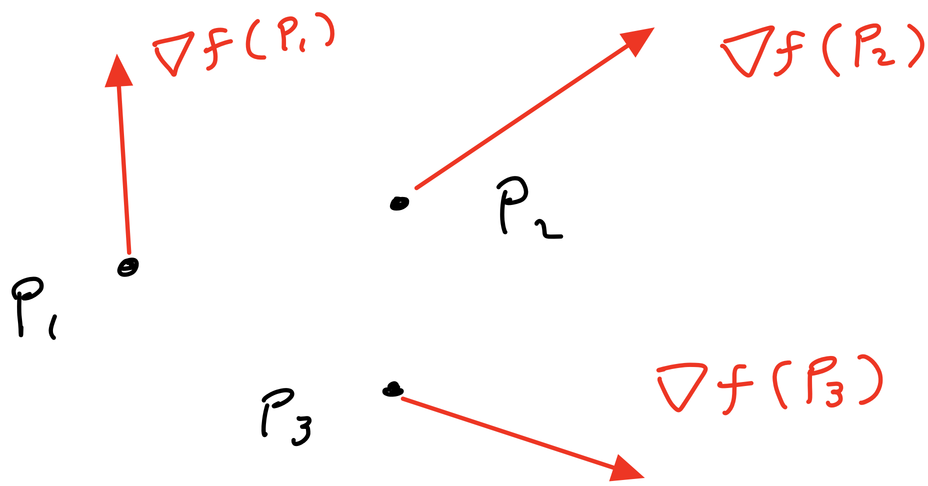
e.g. $f(x,y) = xy$.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x,y) = \langle y, x \rangle$$

To each point P in $\mathbb{R}^n$ the derivative ∇f associates a vector.

Picture:



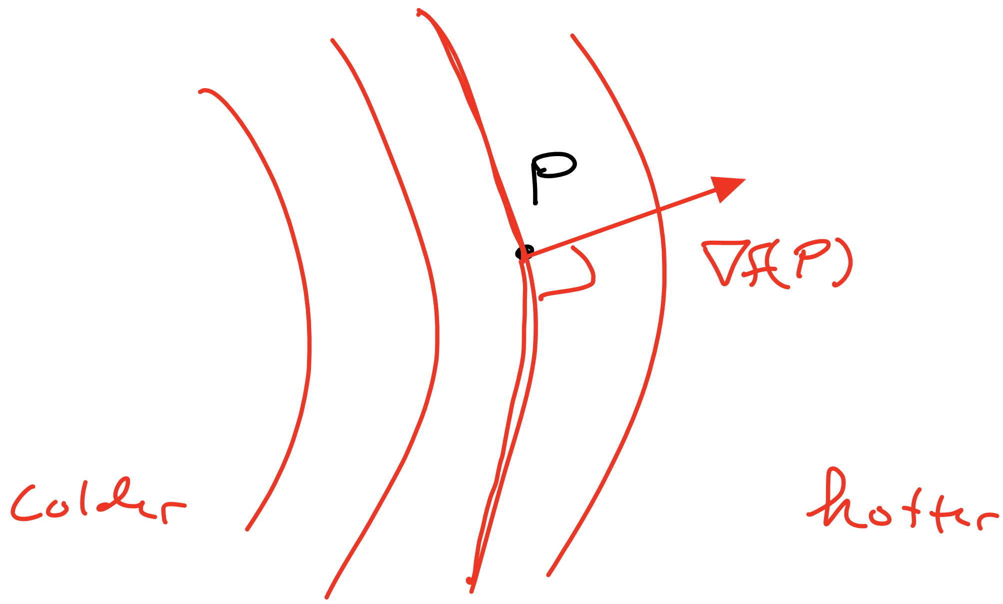
Can think of $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ as a "vector field".

So

"∇(scalar field) = vector field."

Meaning: At a point P in $\mathbb{R}^n$, the vector $\nabla f(P)$ tells us the direction where f is increasing fastest.

e.g. f is temperature



curves of constant temp.

$\nabla f(P)$ points in the direction of increasing temperature & is $\perp$ to the "level curve" at P.

e.g. $f(x,y) = xy$. defined at any point

$\nabla f(x,y) = \langle y, x \rangle.$

pick this point

Consider point $P = (3,1)$.

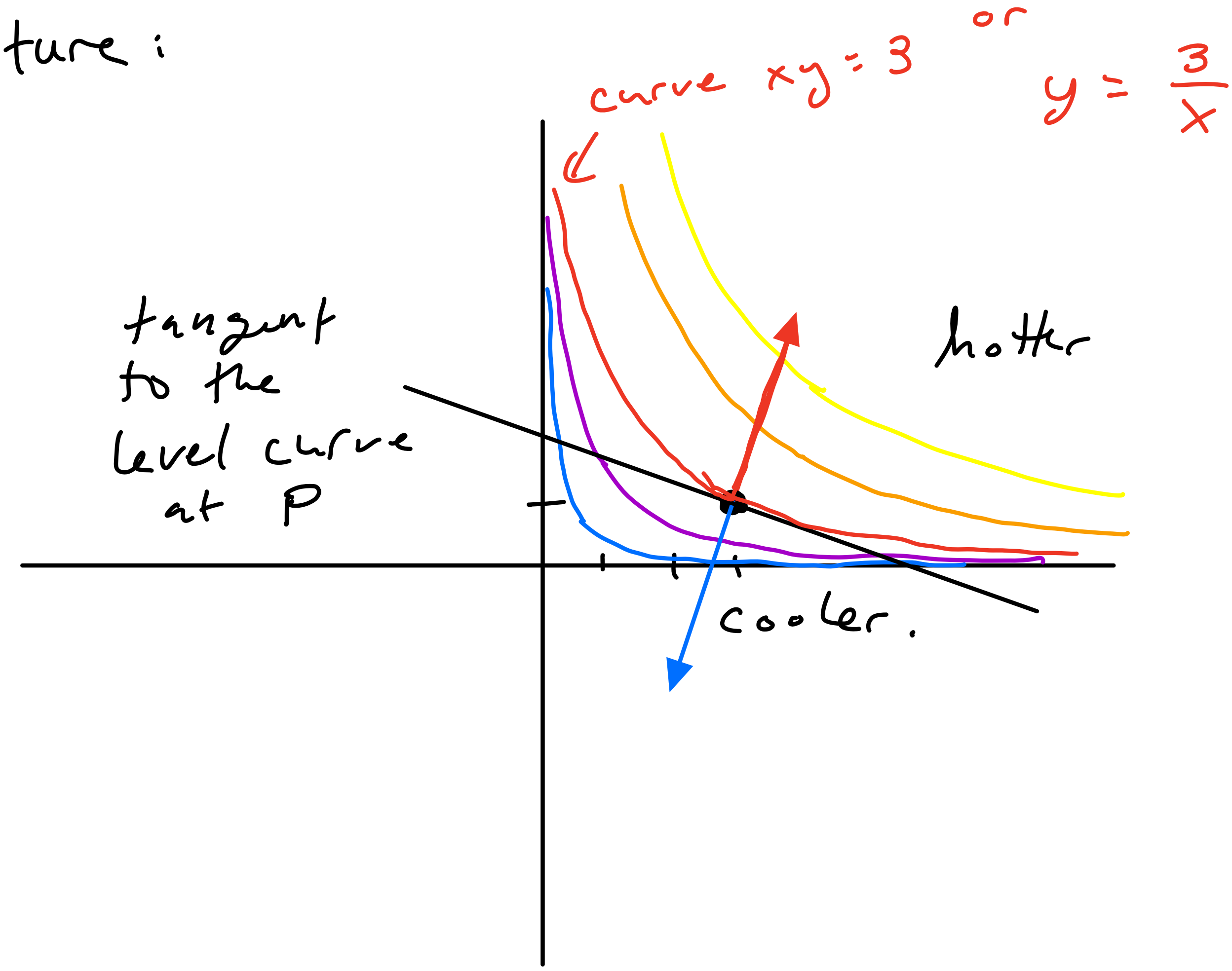
$f(P) = 3 \cdot 1 = 3.$

" temperature at P is 3 ".

Temperature increases fastest in the direction $\nabla f(P) =$

$\nabla f(3,1) = \langle 1, 3 \rangle$

Picture:



$-\nabla f(P)$ is the direction in which

f decreases most rapidly at P.

The "same story" in 3D.

Say $f(x,y,z) = 5x^2 - 3xy + xyz$

is temperature at point $P = (x,y,z)$.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\partial f/\partial x = 10x - 3y + yz$$

$$\partial f/\partial y = 0 - 3x + xz$$

$$\partial f/\partial z = 0 - 0 + xy$$

$$\nabla f(x,y,z)$$

$$= \langle 10x - 3y + yz, -3x + xz, xy \rangle.$$

e.g. Temperature at $P = (1,1,1)$ is

$$f(1,1,1) = 5(1)^2 - 3(1)(1) + (1)(1)(1)$$

$$= 5 - 3 + 1 = 3$$

and increases most rapidly in the direction

$$\nabla f(1,1,1) = \langle 10-3+1, -3+1, 1 \rangle$$
$$= \langle 8, -2, 1 \rangle.$$

Why does it work?

KEY: Multivariable Chain Rule.

Recall the good old chain rule.

$$f : \mathbb{R} \to \mathbb{R}$$
$$g : \mathbb{R} \to \mathbb{R}.$$

$$(f \circ g)(t) = f(g(t)).$$

$$(f \circ g)'(t) = f'(g(t)) \cdot g'(t)$$
$$= (f' \circ g)(t) \cdot g'(t)$$

In Chapter 3 we had a new version.

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

$$f : \mathbb{R} \to \mathbb{R}$$

$$(\vec{r} \circ f)(t) = \vec{r}(f(t)).$$

$$(\vec{r} \circ f)'(t) = \underbrace{\vec{r}\,'(f(t))}_{\text{vector}} \cdot \underbrace{f'(t)}_{\text{scalar.}}$$

Now we have something new.

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

Say $\vec{r}(t)$ is a paraterized path in $\mathbb{R}^n$, say f is temperature. So $f(\vec{r}(t))$ is the temperature we feel at time t. We can think of this as a composition:

$$(f \circ \vec{r})(t) = f(\vec{r}(t))$$

$$f \circ \vec{r} : \mathbb{R} \to \mathbb{R}.$$

$$t \mapsto \text{our temp. at time } t$$

as we travel the curve.

Question: What temperature change do we feel?

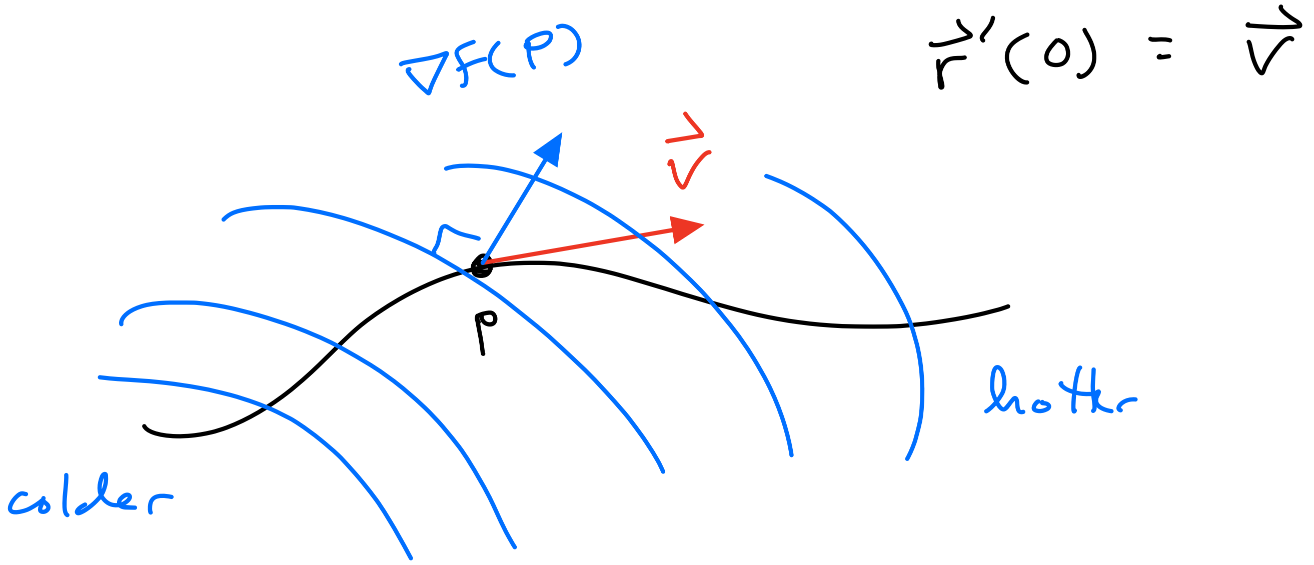
$$(f \circ \vec{r})'(t) = \;?$$

Theorem: dot product.

$$\underbrace{(f \circ \vec{r})'(t)}_{\text{scalar}} = \underbrace{\nabla f(\vec{r}(t))}_{\text{vector}} \cdot \underbrace{\vec{r}\,'(t)}_{\text{vector}}$$

e.g. The rate of change of f at the point P in the direction of vector $\vec{v}$. Suppose $\vec{r}(0) = P$

$$\vec{r}\,'(0) = \vec{v}$$



Our rate of change of temp at time $t = 0$ is

$$(f \circ \vec{r})'(0)$$

$$= \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

$$= \nabla f(P) \cdot \vec{v}.$$

$$= \underbrace{(\text{gradient}) \cdot (\text{velocity})}_{\text{maximized when our velocity points in direction of gradient.}}.$$

Another Consequence: The gradient vector at P is $\perp$ to the level set of f at P.

e.g. Consider $f(x,y,z) = (2x)^2 + y^2 + z^2$.

Temperature at point $P = (2,3,5)$ is

$f(2,3,5) = (2 \cdot 2)^2 + 3^2 + 5^2 = 50.$

Gradient vector is

$$\nabla f(x,y,z) = \langle 8x, 2y, 2z \rangle.$$

$$\nabla f(2,3,5) = \langle 16, 6, 10 \rangle.$$

The level set $f(x,y,z) = 50$ is the set of points where temperature $= 50$.

$$(2x)^2 + y^2 + z^2 = 50.$$

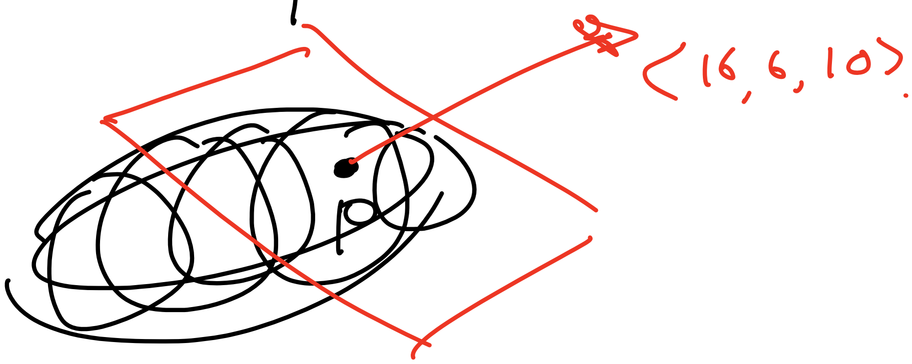
What kind of shape?

Fixed $x \rightsquigarrow$ circle in y, z.

Fixed $y \rightsquigarrow$ ellipse in x, z

Fixed $z \longrightarrow$ ellipse in x, y.

Called an "ellipsoid"



Theorem: $\langle 16, 6, 10 \rangle$ is normal to the tangent plane to the level surface at $P = (2, 3, 5)$.

Conclusion:

Equation of tangent plane to ellipsoid $(2x)^2 + y^2 + z^2 = 50$ at the point $(2, 3, 5)$ is

$$\nabla f(2,3,5) \cdot \langle x-2, y-3, z-5 \rangle = 0.$$

$$\langle 16, 6, 10 \rangle \cdot \langle x-2, y-3, z-5 \rangle = 0$$

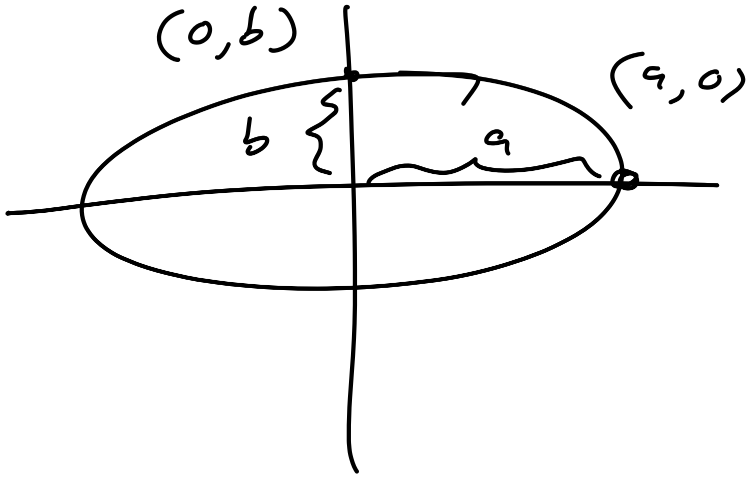
$$16(x-2) + 6(y-3) + 10(z-5) = 0$$

$$\vdots$$

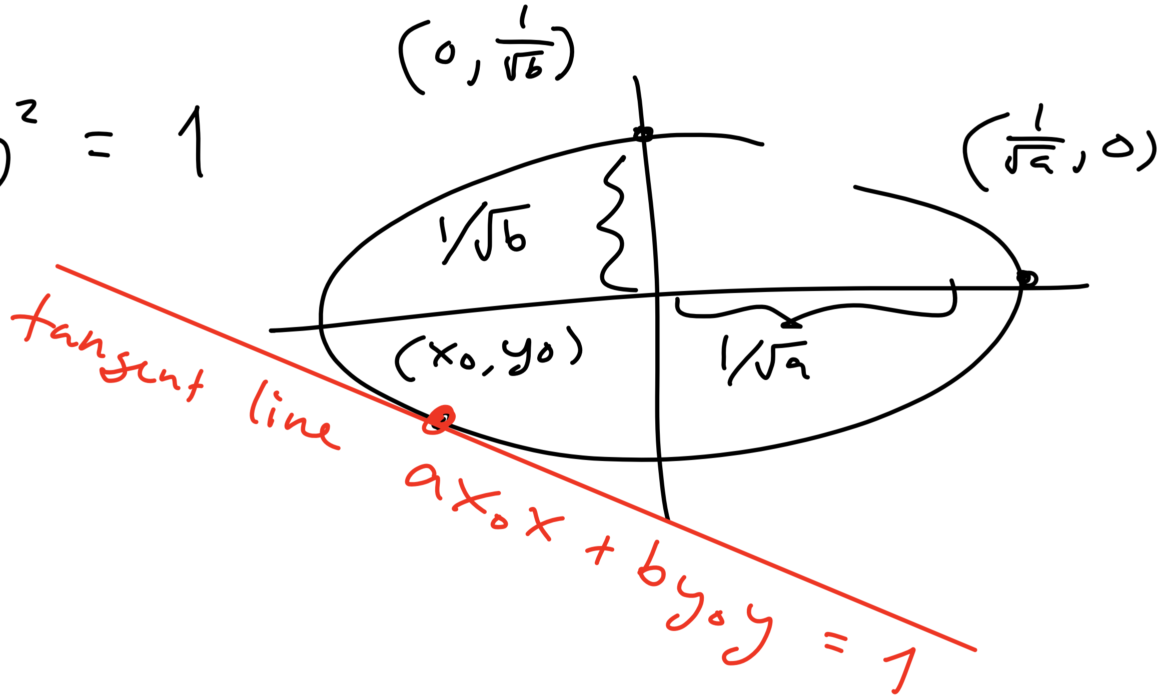
$$8x + 3y + 5z = 50.$$

HW 3 Problem 1:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



$$ax^2 + by^2 = 1$$



HW 3 due Friday.

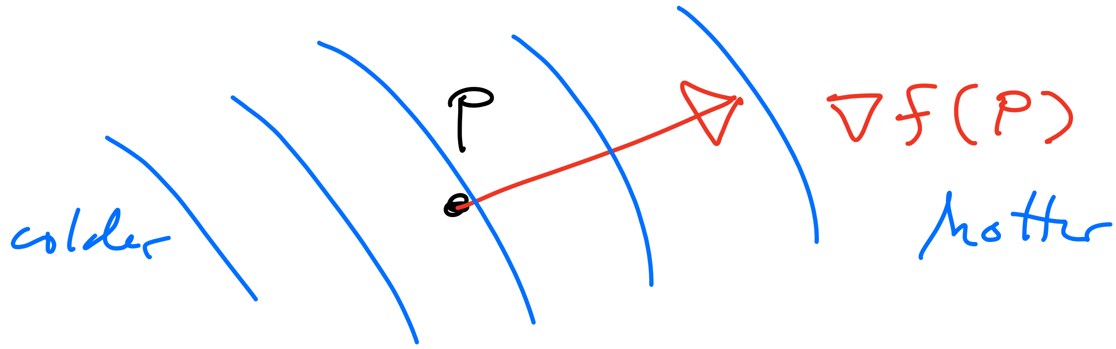
Recall: A scalar field is a function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Assigns scalar $f(P)$ to each point P.

Derivative ∇f assigns a vector

$\nabla f(P)$ to each point P



$f(P)$ = temperature at P

$\nabla f(P)$ = direction of greatest increase of temperature.

Definition:

$f(x_1, \dots, x_n)$

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \dots, \frac{df}{dx_n} \right\rangle.$$

Gradient is $\perp$ to the level curves (curves of constant temperature).

Why?

Multivariable Chain Rule:

$f(P)$ is temperature at point P.

You travel path $\vec{r}(t)$.

Your temperature at time t is

$$T(t) = f(\vec{r}(t)).$$

Your rate of change of temperature at time t is

$$\underbrace{T'(t)}_{\text{scalar}} = \underbrace{\nabla f(\vec{r}(t))}_{\text{vector}} \bullet \underbrace{\vec{r}\,'(t)}_{\text{vector}}$$

dot product.

OR

$$(f \circ \vec{r})'(t) = \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t)$$

composition

$$= (\nabla f \circ \vec{r})(t) \bullet \vec{r}\,'(t)$$

composition of functions

dot product.

$$(f \circ \vec{r})' = (\nabla f \circ \vec{r}) \bullet \vec{r}\,'$$

Consequence: Suppose you travel on a level curve / level surface:

$$f(\vec{r}(t)) = \text{constant}.$$

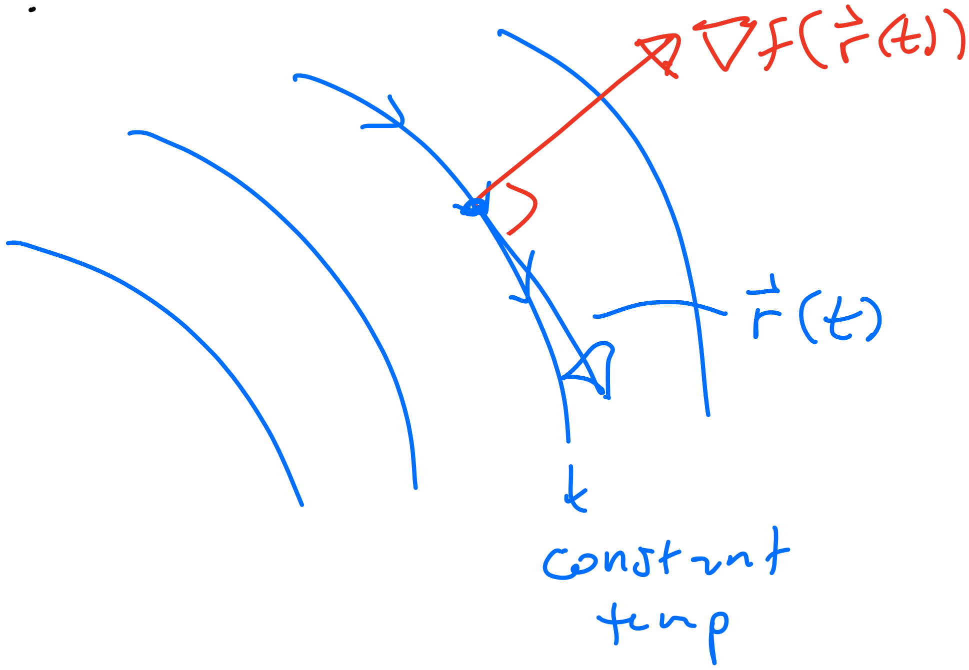
Then

$$\frac{d}{dt}\left[f(\vec{r}(t)) \right] = 0$$

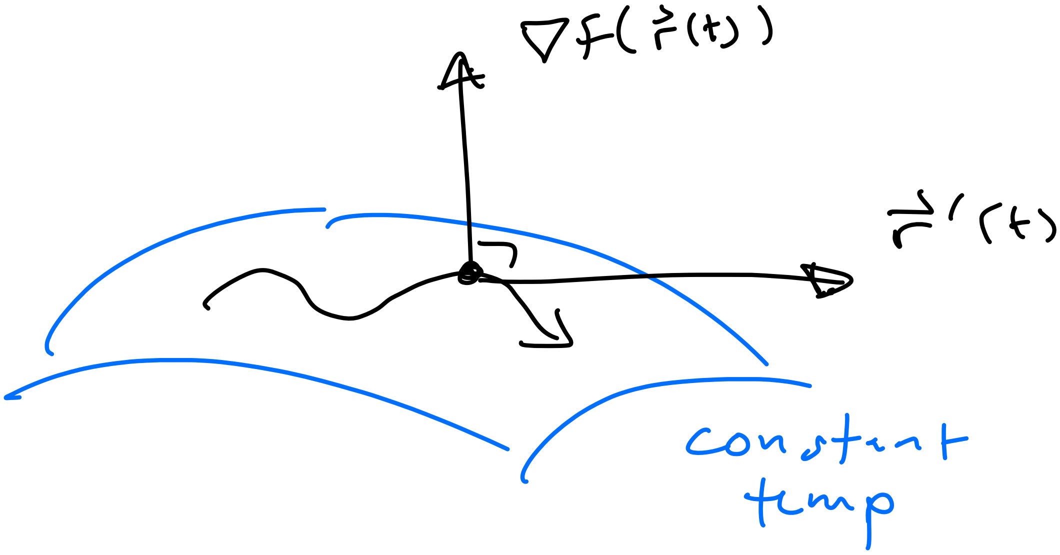
$$\underbrace{\nabla f(\vec{r}(t))} \cdot \underbrace{\vec{r}\,'(t)} = 0$$

perpendicular vectors.

Picture:



Picture in 3D: level surfaces



So $\nabla f(P)$ is $\perp$ to the level surface through any point P.

Example: Consider scalar field

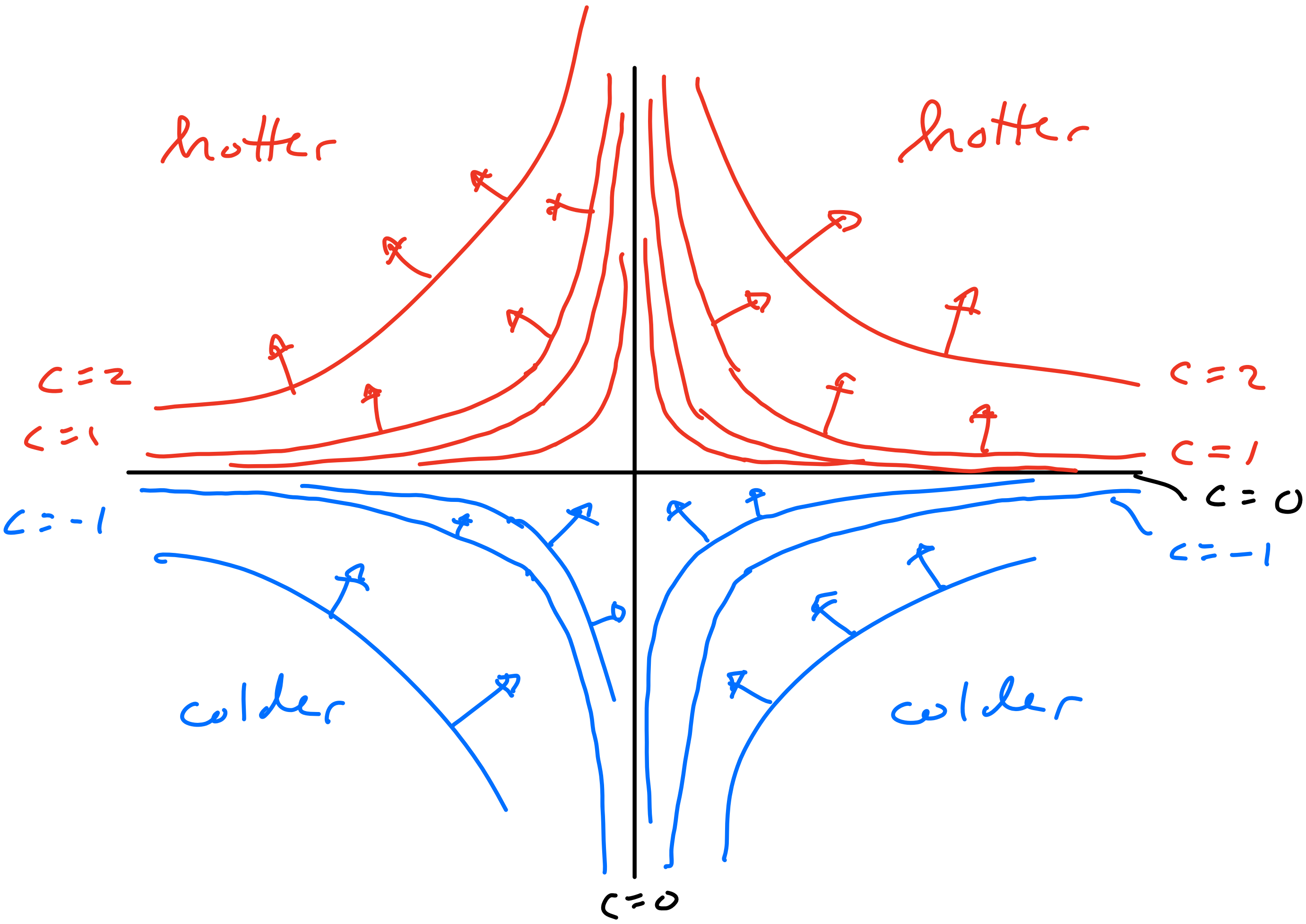
$$f(x,y) = x^2 y.$$

Level curves $f(x,y) = c$ (constant)

$$x^2 y = c$$

$$y = \frac{c}{x^2}$$

What do these curves look like?



Suppose we travel along the curve

$$\vec{r}(t) = \langle t, 2-t^2 \rangle.$$

Our temperature at time t is

$$T(t) = f(\vec{r}(t))$$
$$= f(t, 2-t^2)$$
$$= (t)^2(2-t^2)$$
$$= 2t^2 - t^4$$

When is our temperature maximized or minimized?

$$T'(t) = 0$$

$$4t - 4t^3 = 0$$

$$4t(1-t^2) = 0$$

$$\Rightarrow \quad t = 0 \quad \text{or} \quad 1 - t^2 = 0$$

$$t = \pm 1.$$

Second Derivative:

$$T''(t) = 4 - 12t^2$$

So min at $t = 0$ $\quad [T''(0) > 0]$

max at $t = \pm 1$ $\quad [T''(\pm 1) < 0]$

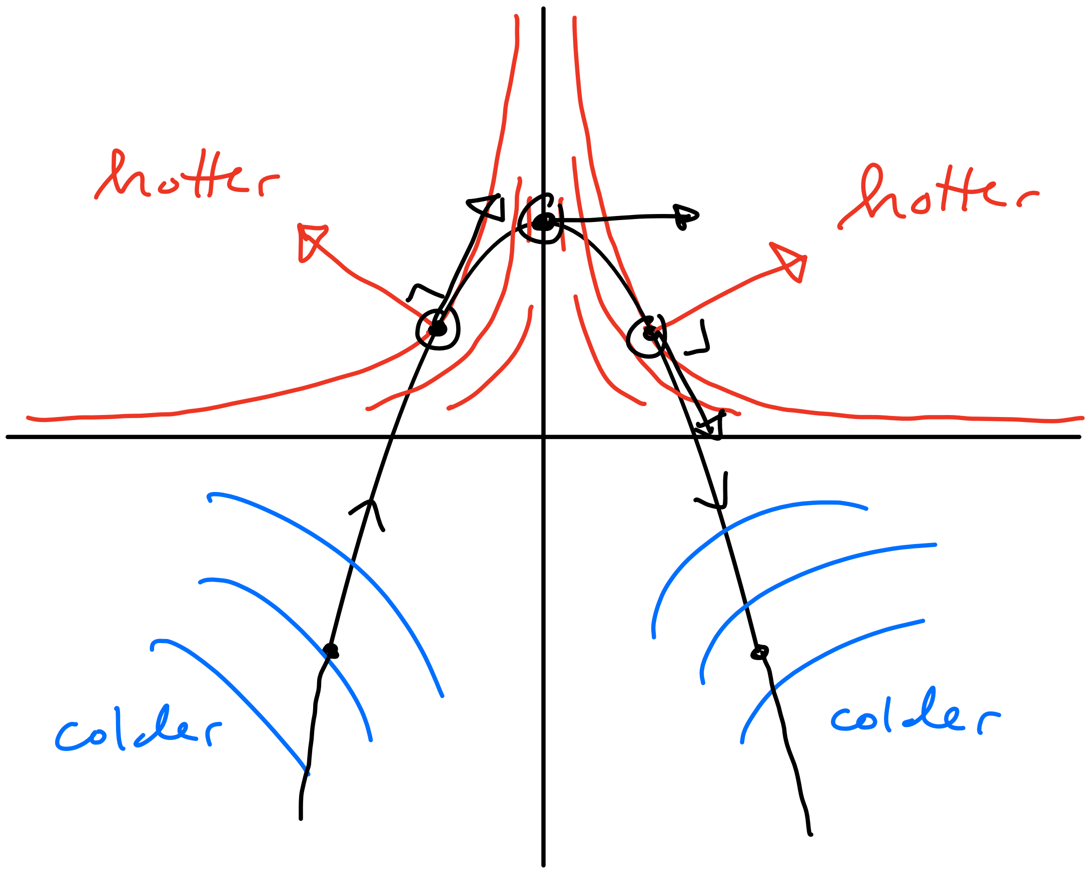
Where is temperature max or min?

$$\vec{r}(0) = \langle 0, 2 \rangle$$

$$\vec{r}(+1) = \langle +1, 2 - (+1)^2 \rangle = \langle 1, 1 \rangle$$

$$\vec{r}(-1) = \langle -1, 2 - (-1)^2 \rangle = \langle -1, 1 \rangle.$$

Picture:



Local maxima happened when our velocity is $\perp$ to gradient.

Indeed, local max $\Rightarrow$ $T'(t) = 0$

$$T'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$0 = \underbrace{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\text{perpendicular!}}$$

$t=0$ a bit diffrent because $\nabla f(\vec{r}(0)) = \langle 0, 0 \rangle$.

Every vector is $\perp$ to $\langle 0,0 \rangle$.

So that case is "degenerate".

Another point of view.

Instead of $\vec{r}(t) = \langle t, 2-t^2 \rangle$,

eliminate t. The parabola is

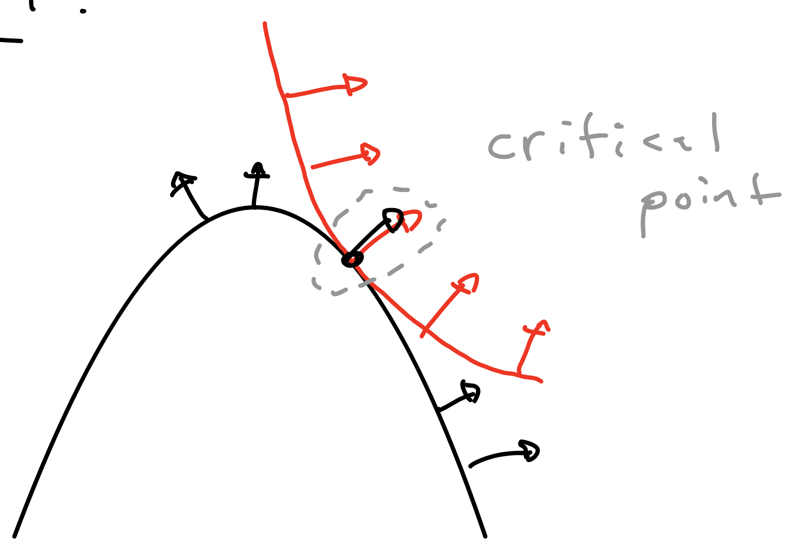
$$y = 2 - x^2$$

$$x^2 + y = 2$$

$$g(x,y) = 2$$

for $g(x,y) = x^2 + y$.

In this language, look for points where $\nabla f(x,y)$ & $\nabla g(x,y)$ are <u>parallel</u>.



This is the method of "Lagrange Multipliers". Calculate:

If $\nabla f(x,y)$ & $\nabla g(x,y)$ are parallel then

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

for some scalar λ.

In our case,

$$f(x,y) = x^2 y \quad \text{(temp.)}$$

$$g(x,y) = x^2 + y \quad \text{(defines our parabola)}$$

$$\nabla f(x,y) = \langle 2xy, x^2 \rangle$$

$$\nabla g(x,y) = \langle 2x, 1 \rangle$$

Set $\langle 2xy, x^2 \rangle = \lambda \langle 2x, 1 \rangle$

$$\langle 2xy, x^2 \rangle = \langle 2x\lambda, \lambda \rangle$$

$$\begin{cases} 2xy = 2x\lambda \\ x^2 = \lambda \end{cases}$$

And we are only interested in points on the parabola $y = 2 - x^2$.

So get 3 equations in 3 unknowns:

$$\begin{cases} 2xy = 2x\lambda, & (1) \\ x^2 = \lambda, & (2) \\ y = 2 - x^2. & (3) \end{cases}$$

In general, VERY HARD to solve.

But this "textbook problem" is not bad.

If $x = 0$ then $y = 2$.

And $(x,y) = (0,2)$ is a solution.

If $x \neq 0$ then

(1): $2\cancel{x}y = 2\cancel{x}\lambda$

$y = \lambda$

(2): $x^2 = \lambda$

$x^2 = y$

(3): $y = 2 - x^2$.

② & ③ : $$x^2 = 2 - x^2$$
$$2x^2 = 2$$
$$x^2 = 1$$
$$x = \pm 1.$$

So two more critical points

$$(x, y) = (+1, +1) \text{ or } (-1, +1).$$

Same solution as before

Three critical points

$$(0,2), \quad (1,1), \quad (-1,1)$$

(0,2)	(1,1)	(-1,1)
min of f	max of f	max of f

Lagrange Multipliers in General:

Maximize $f(x_1, x_2, \dots, x_n)$

subject to constraint

$$g(x_1, x_2, \dots, x_n) = K$$

Solution: Find all critical

points $(x_1, \dots, x_n)$ such that

$$\begin{cases} \nabla f(x_1, \dots, x_n) = \lambda \nabla g(x_1, \dots, x_n) \\ g(x_1, \dots, x_n) = K \end{cases}$$

In general "impossible" to solve exactly, so use a computer to get numerical solutions.

Linear Approximation:

Another point of view on the chain rule.

$$\frac{d}{dt}\left[f(\vec{r}(t)) \right] = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Let's write

$$f(x, y, z)$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle.$$

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

Then the chain rule says:

$$\frac{dF}{dt} = \nabla f \cdot \vec{r}\,'$$

$$= \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

$$\boxed{\frac{dF}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt} + \frac{df}{dz}\cdot\frac{dz}{dt}}$$

PURE ALGEBRA !

2D version:

$f(x,y)$ function of x & y.

$x(t)$, $y(t)$ functions of t.

$$\frac{dF}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt}.$$

Intuition: $$\frac{df}{\cancel{dx}}\cdot\frac{\cancel{dx}}{dt} \approx \frac{df}{dt}$$

NOT correct math!

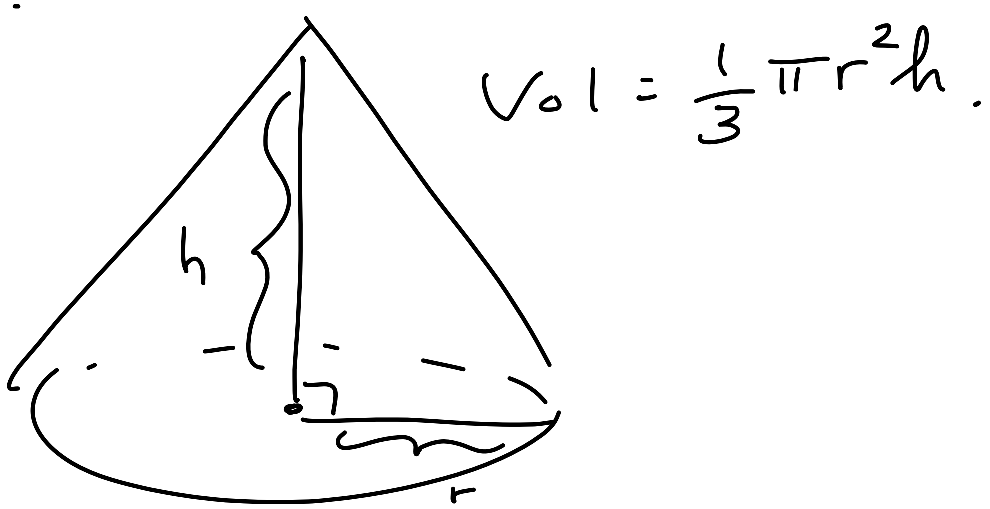
Application :

Consider a right circular cone with height h & radius r.

Volume is a function of h & r :

$$V(r,h) = \frac{1}{3}\pi r^2 h$$

Picture :



Suppose h & r change with time : $h(t)$, $r(t)$.

Then the volume changes with time :

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} + \frac{dV}{dh} \cdot \frac{dh}{dt}$$

To simplify terminology, sometimes

we write $V_t = dV/dt$

$V_r = dV/dr$

$V_h = dV/dh.$

$$V_t = V_r \cdot \frac{dr}{dt} + V_h \cdot \frac{dh}{dt}$$

Have $V_r = \frac{1}{3}\pi 2rh$

$V_h = \frac{1}{3}\pi r^2$

So

$$\frac{dV}{dt} = \frac{1}{3}\pi 2rh \cdot \frac{dr}{dt} + \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt}$$

But maybe it's not changing with time; it's changing for some other reason. So let's just say

$$dV = \frac{1}{3}\pi 2rh \cdot dr + \frac{1}{3}\pi r^2 \cdot dh$$

tiny change in V

related to tiny changes in r & h.

Application : Error estimation.

Measure the radius & height :

$$r = 120 \pm 1.8 \text{ in}$$
$$h = 140 \pm 2.5 \text{ in}$$

Then $V = \frac{1}{3}\pi r^2 h \pm dV$

$$V = 2{,}111{,}150 \pm dV$$

how big is the "error" ?

Errors are related by chain rule :

$$dV = \frac{dV}{dr} \cdot dr + \frac{dV}{dh} \cdot dh$$

$$dV = \frac{1}{3}\pi 2rh \cdot dr + \frac{1}{3}\pi r^2 \cdot dh.$$

$$dV = \frac{1}{3}\pi 2(120)(140) \cdot (1.8)$$
$$+ \frac{1}{3}\pi (120)^2 \cdot (2.5)$$

$$= 101{,}033 \text{ in}^3.$$

We conclude that

$$V = 2{,}111{,}150 \pm 101{,}033 \text{ in}^3$$

$$= 2.11 \pm 0.1 \text{ million in}^3.$$

HW3 Problem 3:

Convert θ to radians.

HW3 Problem 2:

For any resonable function $f(x,y)$ we will have

$$f_{xy} = f_{yx}$$

$$\frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{df}{dy}\right)$$

There are some pathological counterexamples, but we can ignore them.

Hint: $f(x,y)$, $x(r,\theta)$, $y(r,\theta)$.

f is (indirectly) a function of r & θ. Like to compute derivatives

$$f_r, f_\theta, f_{rr}, f_{r\theta}, f_{\theta\theta}.$$

For these we need chain rule.

$$f_r = f_x \cdot x_r + f_y \cdot y_r$$

$$f_{rr} = \frac{d}{dr}\left(f_x \cdot x_r + f_y \cdot y_r \right)$$

$$= \underbrace{\frac{d}{dr}\left(f_x \cdot x_r\right)}_{\text{product rule}} + \underbrace{\frac{d}{dr}\left(f_y \cdot y_r\right)}_{\text{product rule}}$$

$$= f_{xr} \cdot x_r + f_x \cdot x_{rr}$$

$$+ f_{yr} \cdot y_r + f_y \cdot y_{rr}$$

Also need to simplify f_{xr}, f_{yr}.

Well, $f_x(x,y)$ is a function of 2 variables x & y, so

$$f_{xr} = f_{xx} \cdot x_r + f_{xy} \cdot y_r .$$

$$\frac{df_x}{dr} = \frac{df_x}{dx} \cdot \frac{dx}{dr} + \frac{df_x}{dy} \cdot \frac{dy}{dr} .$$

More on Linear Approximation.

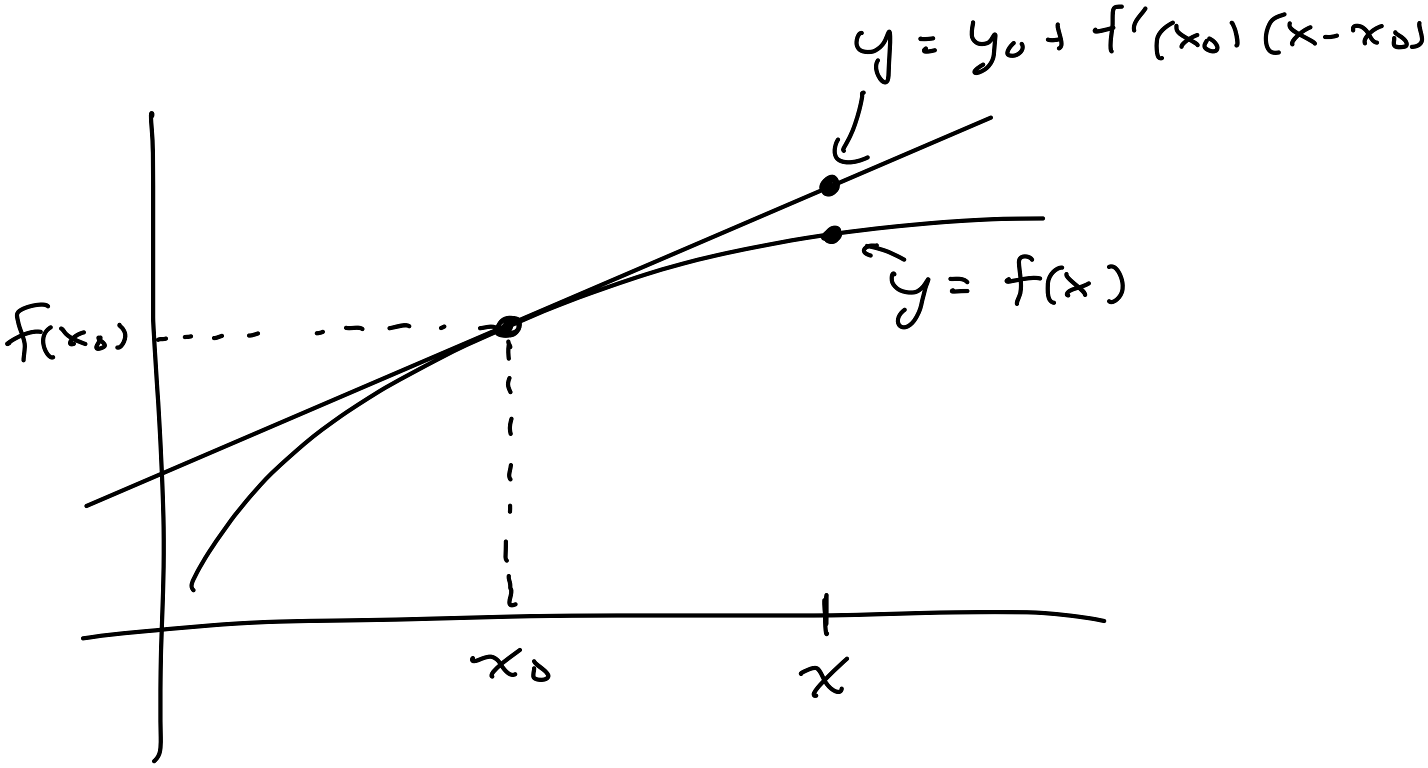
Given $f: \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ we have a Taylor expansion of $f(x)$ near $x = x_0$:

$$
\begin{aligned}
f(x) = f(x_0) &+ f'(x_0)(x - x_0) \\
&+ \frac{1}{2} f''(x_0)(x - x_0)^2 \\
&+ \frac{1}{6} f'''(x_0)(x - x_0)^3 \\
&\vdots \\
&+ \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \\
&\vdots
\end{aligned}
$$

Get an approximation by stopping the series early. Linear approx:

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)}$$

Picture:



Equation of the tangent line.

slope $f'(x_0)$, point (x_0, y_0).

Point-Slope form:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$y = y_0 + f'(x_0)(x - x_0)$$

Why do we care?

$$f(x) \approx \underset{f(x_0)}{\cancel{y_0}} + f'(x_0)(x - x_0)$$

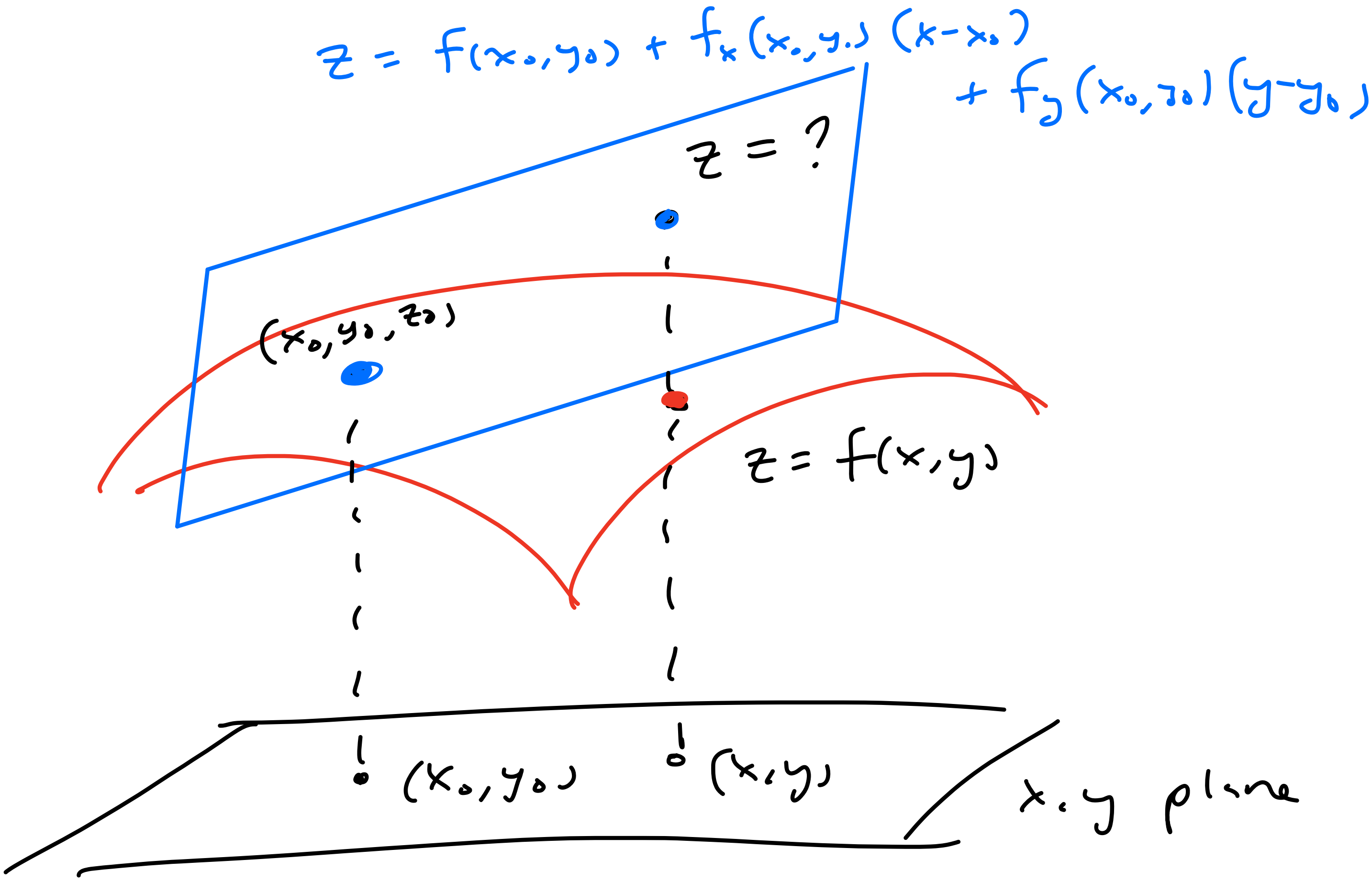
EASIER TO CALCULATE!

Function of 2 variables $f(x,y)$ has a Taylor expansion near $(x,y) = (x_0, y_0)$:

$$f(x,y) = f(x_0,y_0)$$
$$+ f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$+ \frac{1}{2}\Big[f_{xx}(x_0,y_0)(x-x_0)^2$$
$$+ 2 f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$
$$+ f_{yy}(x_0,y_0)(y-y_0)^2 \Big]$$
$$+ \frac{1}{6}\Big[\text{stuff like}$$
$$f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 \cdots \Big]$$
$$\vdots$$

Cut it off to get linear approx:

$$\underset{z}{f(x,y)} \approx \underset{z_0}{f(x_0,y_0)} + f_x(x_0,y_0)(x-x_0)$$
$$+ f_y(x_0,y_0)(y-y_0).$$



$$z_0 = f(x_0,y_0)$$

Equation of Tangent Plane:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Linear Approx:

$$z \approx z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Language: $y = f(x)$

$$y - y_0 \approx f'(x_0)(x - x_0)$$

$$\Delta y \approx f'(x_0)\,\Delta x$$

$$\Delta y \approx \frac{dy}{dx}\,\Delta x$$

$$dy = \frac{dy}{dx} \cdot dx$$

Chain Rule

$$\left(\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \right)$$

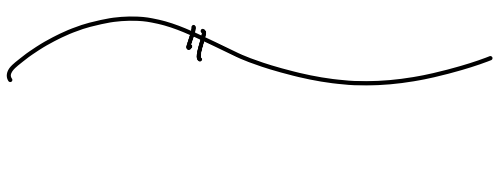
$$z = f(x, y)$$

$$z - z_0 \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

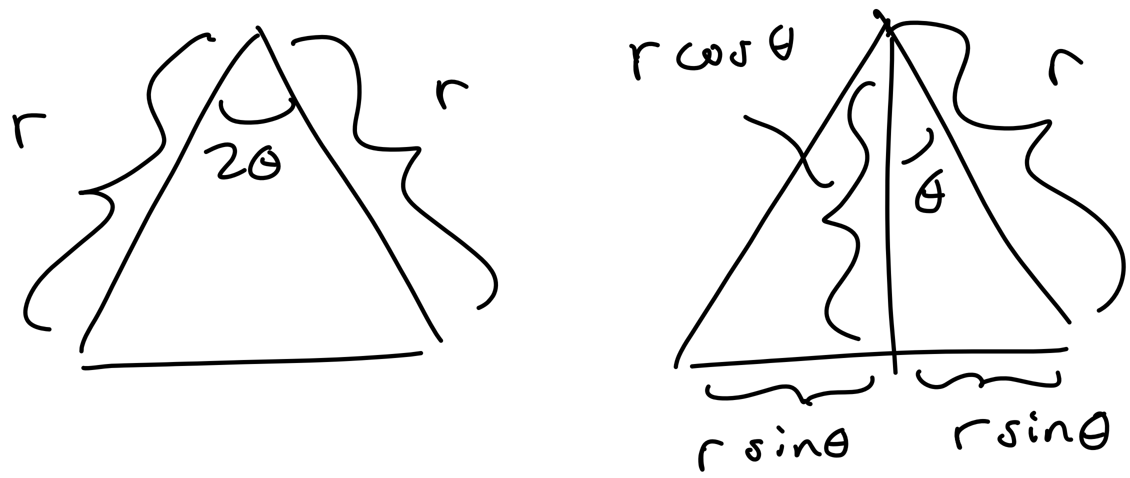
$$\Delta z \approx \frac{df}{dx} \cdot \Delta x + \frac{df}{dy} \cdot \Delta y .$$

$$dz = \frac{df}{dx}\,dx + \frac{df}{dy}\,dy .$$

Chain Rule!



Example: Estimate area of isoceles triangle



$$A(r,\theta) = \frac{1}{2}\,\text{base}\cdot\text{height}$$

$$= \frac{1}{2}(2r\sin\theta)(r\cos\theta)$$

$$= r^2 \sin\theta\cos\theta$$

$$= \frac{r^2}{2}\sin(2\theta)$$

$$A_r = r\sin(2\theta)$$

$$A_\theta = \frac{r^2}{2}\cos(2\theta)\cdot 2$$

$$= r^2\cos(2\theta).$$

Linear Approximation

$$dA = A_r\,dr + A_\theta\,d\theta$$

$$= r\sin(2\theta)\,dr + r^2\cos(2\theta)\,d\theta.$$

OR. Near r_0, θ_0,

$$A(r,\theta) - A(r_0,\theta_0) \approx$$

$$+\ r_0\sin(2\theta_0)(r - r_0)$$

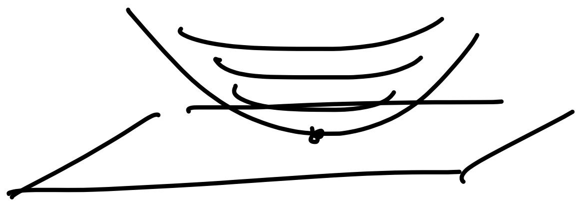
$$+\ r_0^2\cos(2\theta_0)(\theta - \theta_0)$$

Unconstrained Optimization.

Find local max/min of a scalar field $f(x,y)$.

If we look at the graph $z = f(x,y)$ these are places where the tangent plane is horizontal:

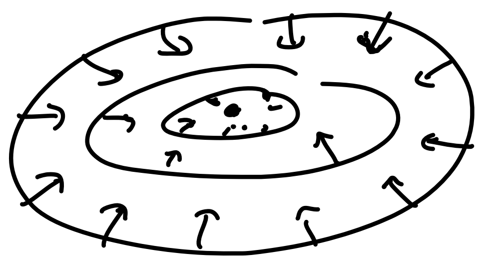
Local Max



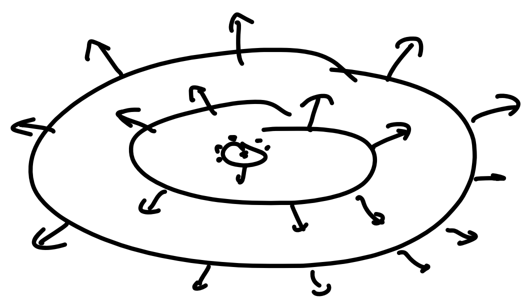
This happens when $\nabla f(x,y)$ vanishes, i.e., equals $\langle 0,0\rangle$.

In 2D:

Local Max



Local Min



Example: $f(x,y) = x^2 + y^2$.

$\nabla f(x,y) = \langle 2x, 2y\rangle = \langle 0,0\rangle$.

"Critical point" $(x,y) = (0,0)$.

Turns out to be a min.

(Parabolic shaped Valley).

Example: $f(x,y) = xy$.

$$\nabla f(x,y) = \langle y, x \rangle = \langle 0,0 \rangle$$

critical point $(x,y) = (0,0)$

Min or Max? NO!

It's a "saddle point".

[see Geogebra]

Example: $f(x,y) = x^2 y$

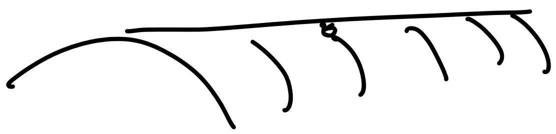
$$\nabla f(x,y) = \langle 2xy, x^2 \rangle = \langle 0,0 \rangle.$$

Every point $\langle 0, y \rangle$ is critical!

These turn out to be degenerate critical points.

[See Geogebra].

Summary:



degenerate
(flat in some
direction)

The "second derivative test"
lets us distinguish between these.

Hessian Matrix of $f(x,y)$:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Hessian determinant

$$\det(Hf) = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yx}$$

$$= f_{xx} \cdot f_{yy} - f_{xy}^2.$$

Theorem:

Let (x_0, y_0) be critical:

$$\nabla f(x_0, y_0) = \langle 0, 0 \rangle$$

i.e. $f_x(x_0, y_0) = 0$ & $f_y(x_0, y_0) = 0$

Then:

- If $\det(Hf)(x_0, y_0) < 0$ then (x_0, y_0) is a <u>saddle</u> point.

- If $\det(Hf)(x_0, y_0) = 0$ called "degenerate". [Too flat for this test to work ...]

- If $\det(Hf)(x_0, y_0) > 0$ then (x_0, y_0) is local max or min
    - Max when $f_{xx}(x_0, y_0) < 0$
    - Min when $f_{xx}(x_0, y_0) > 0$

Example: $f(x,y) = x^2 - x^4 - y^2 + 1$

$$\nabla f(x,y) = \langle 2x - 4x^3, -2y \rangle = \langle 0, 0 \rangle$$

$$\begin{cases} 2x - 4x^3 = 0 \\ -2y = 0 \longrightarrow y = 0 \end{cases}$$

ALWAYS

$$2x(1-2x^2) = 0$$

$$x = 0 \quad \text{or} \quad x = \pm 1/\sqrt{2}.$$

3 critical points:

$$(0,0), \left(\tfrac{1}{\sqrt{2}}, 0\right), \left(-\tfrac{1}{\sqrt{2}}, 0\right)$$

$$f_{xx} = (2x - 4x^3)_x = 2 - 12x^2$$

$$\left.\begin{aligned} f_{xy} &= (2x - 4x^3)_y = 0 \\ f_{yx} &= (-2y)_x = 0 \end{aligned}\right\} \text{SAME} \ :)$$

$$f_{yy} = (-2y)_y = -2$$

Hessian Matrix

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2-12x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\det(Hf) = (2-12x^2)(-2) - 0$$

$$= -2 + 12x^2.$$

Evaluate $\det(Hf)$ at our critical points:

- $\det(Hf)(0,0) = -2 + 12 \cdot 0^2$
  $= -2 < 0$.
  SADDLE.

- $\det(Hf)\left(\frac{1}{\sqrt{2}}, 0\right) = -2 + 12\left(\frac{1}{\sqrt{2}}\right)^2$
  $= -2 + 6 = 4 > 0$

- $\det(HF)\left(-\frac{1}{\sqrt{2}}, 0\right) = -2 + 12\left(-\frac{1}{\sqrt{2}}\right)^2$
  $= -2 + 6 = 4 > 0$

Local max or min . . .

To see which we look at f_{xx}.

$f_{xx}(x,y) = 2 - 12x^2$.

$f_{xx}\left(\frac{1}{\sqrt{2}}, 0\right) = 2 - 12\left(\frac{1}{\sqrt{2}}\right)^2$

$= 2 - 6 = -4 < 0$

MAX!

$f_{xx}\left(-\frac{1}{\sqrt{2}}, 0\right) = 2 - 12\left(-\frac{1}{\sqrt{2}}\right)^2$

$= 2 - 6 = -4 < 0$

MAX!

CALC I