Chap 1: Parametrized paths & Arc length

Now: Chapter 2, vectors, equations of lines & planes.

Equation of a line can be written in several different ways:

slope, intercept $\qquad y = mx + b$

slope, point $\qquad m = (y-y_0)/(x-x_0)$

two points (x_0, y_0) & (x_1, y_1) $\qquad m = (y_2-y_1)/(x_2-x_1)$

$$\frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)}$$

$$(y-y_1)(x_2-x_1) = (x-x_1)(y_2-y_1)$$

In this course we use the form

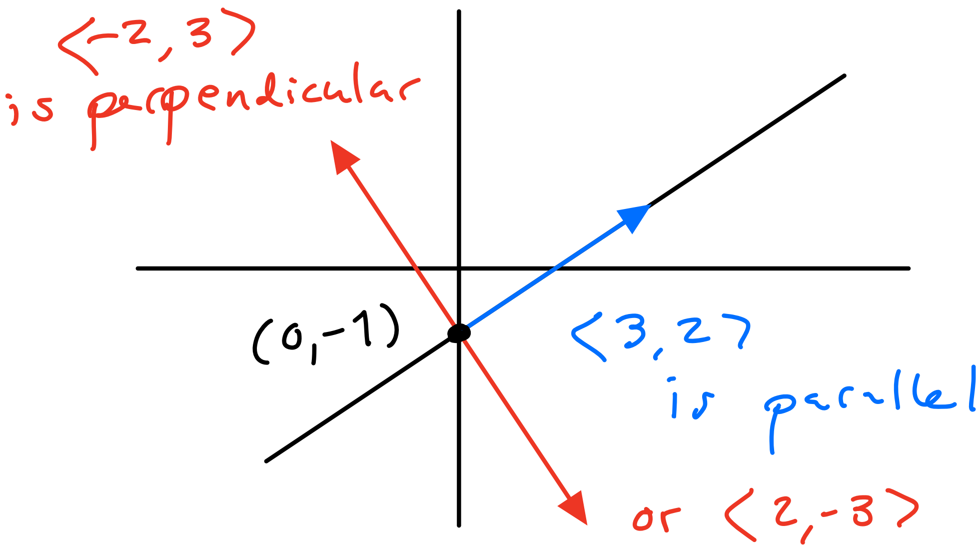
$$ax + by = c$$

OR

$$a(x-x_0) + b(y-y_0) = 0$$

This is the line passing through point (x_0, y_0) & $\perp$ to vector $\langle a, b \rangle$.

Example: $y = \frac{2}{3}x - 1$



Convert to point, perpendicular form.

Take $(x_0, y_0) = (0, -1)$.

Need a vector .....

Trick: Find a vector in the line (say $\langle 3, 2 \rangle$ because slope $2/3$), then take the negative reciprocal slope $-3/2$, which suggests

$$\langle a, b \rangle = \langle -2, 3 \rangle \text{ or } \langle 2, -3 \rangle$$

In fact, <u>any</u> scalar multiple of $\langle -2, 3\rangle$ or $\langle 2, -3\rangle$ will work.

Thus we get the equation

$$a(x-x_0) + b(y-y_0) = 0$$

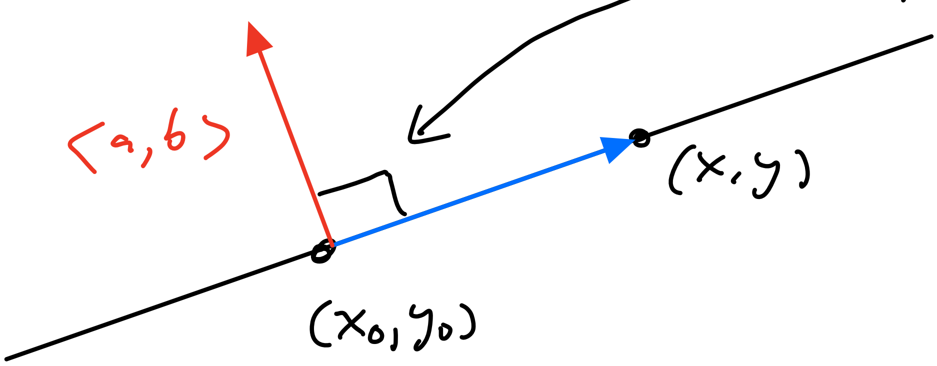
$$2(x-0) - 3(y+1) = 0.$$

$\langle a, b\rangle = \langle 2, -3\rangle$ $\qquad$ $(x_0, y_0) = (0, -1)$.

This equation says that

$$\langle a, b\rangle \cdot \langle x - x_0, y - y_0\rangle = 0$$

vectors are perp.



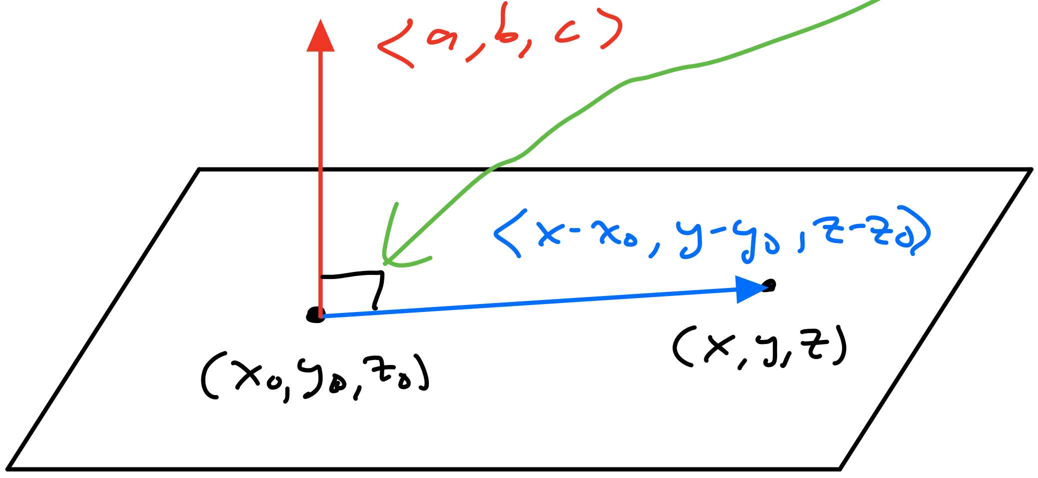
More into 3D:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

$$\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0.$$

perpendicular

Picture: A plane. Given point (x_0,y_0,z_0) in the plane & vector $\langle a,b,c\rangle$ $\perp$ to the plane, any point (x,y,z) in plane must satisfy this equation.



[Perp vector $\langle a,b,c\rangle$ is also called a "normal vector" for the plane, so

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

is the "normal equation."]

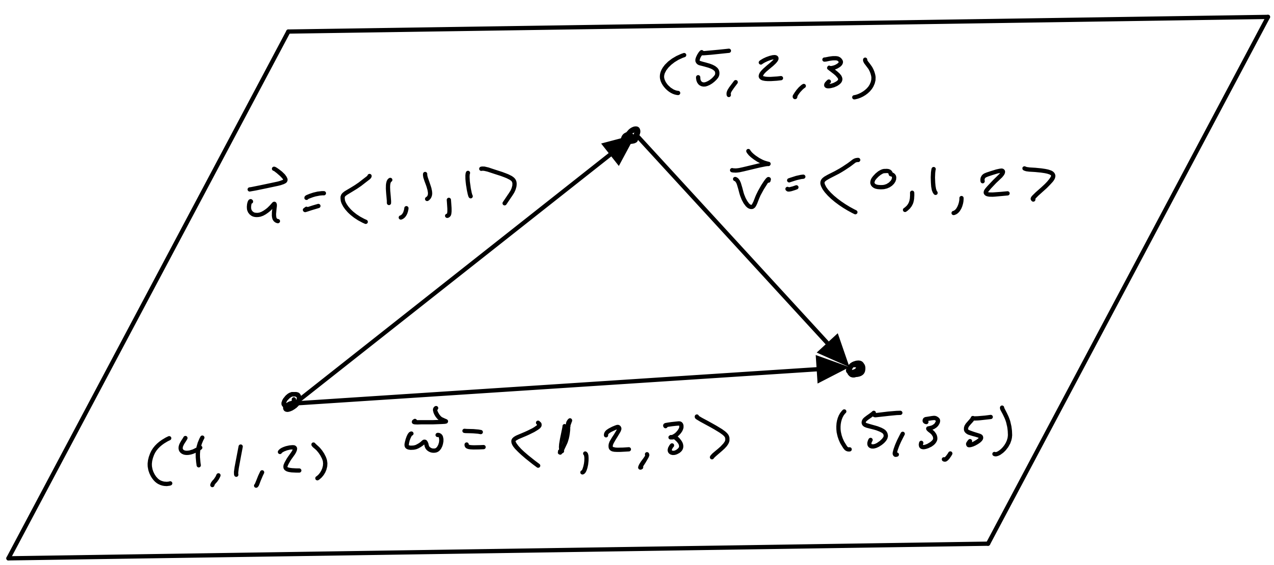
Example: 3 points determine a plane. Find the normal equation of the plane containing points

$P=(4,1,2)$

$Q=(5,2,3)$

$R=(5,3,5)$

Triangle in space



We need to find a normal vector

$\vec{n} = \langle a, b, c \rangle$

pointing $\perp$ to the plane.

There is a TRICK.

Definition of "Cross Product":

Given two vectors in 3D space

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

we define a new vector $\vec{u} \times \vec{v}$ as follows:

$$\vec{u} \times \vec{v} =$$

$$\langle u_2 v_3 - u_3 v_2 \,,\; u_3 v_1 - u_1 v_3 \,,\; u_1 v_2 - u_2 v_1 \rangle$$

Wow, what a mess!

Two Issues:

1. What does it mean?
2. How can we memorize the formula?

① Meaning: $\vec{u} \times \vec{v}$ is simultaneously $\perp$ to $\vec{u}$ & $\vec{v}$.

Check:

$\vec{u} \cdot (\vec{u} \times \vec{v}) =$

$u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1)$

$= u_1 \cancel{u_2} v_3 + u_2 \cancel{u_3} v_1 + u_3 \cancel{u_1} v_2$
$\quad - u_1 \cancel{u_3} v_2 - u_2 \cancel{u_1} v_3 - u_3 \cancel{u_2} v_1$

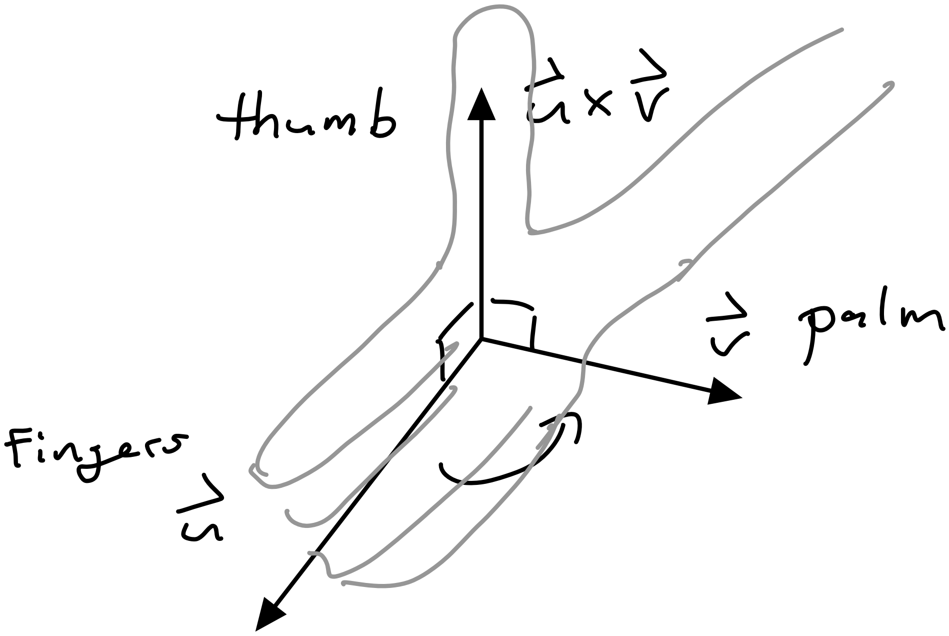
$= 0$ Magic!

So $\vec{u} \perp \vec{u} \times \vec{v}$.

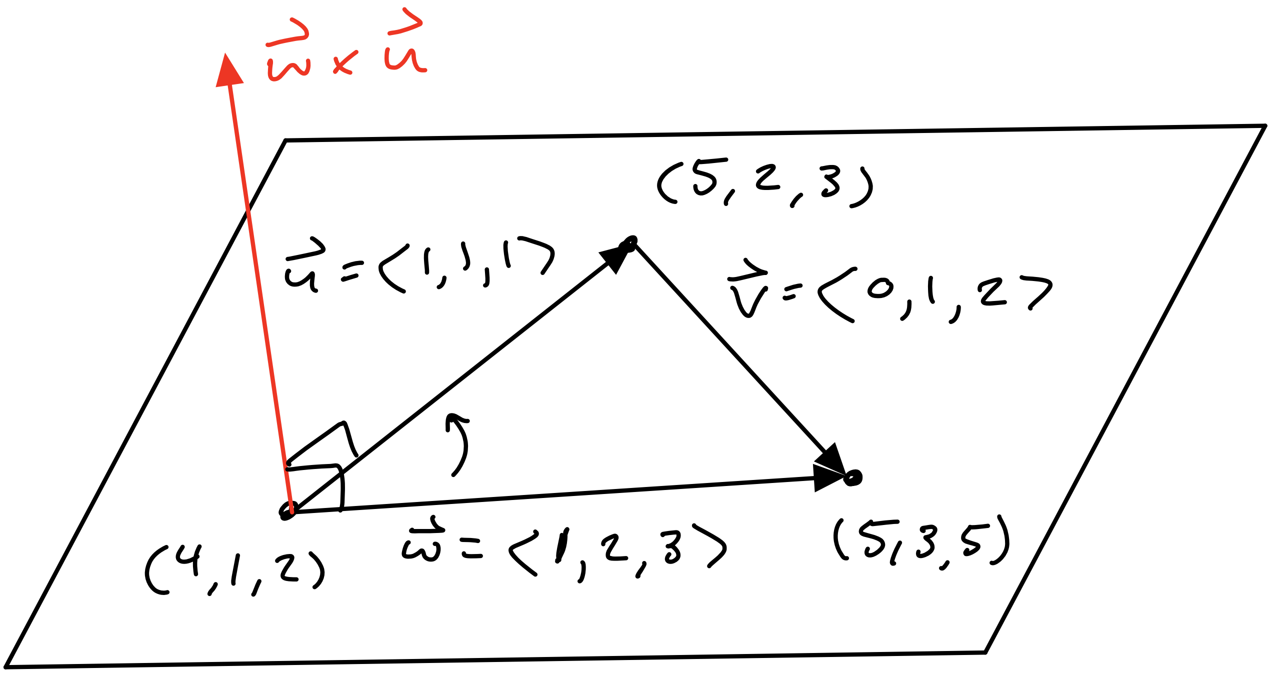
Similar computation shows $\vec{v} \perp \vec{u} \times \vec{v}$.

[The 3D version of the "negative reciprocal slope" trick.]

Picture: Right Hand Rule.



Back to our example:



We have 6 different ways to create a normal vector

$\vec{u} \times \vec{v}$, $\vec{u} \times \vec{w}$, $\vec{v} \times \vec{w}$

$\vec{v} \times \vec{u}$, $\vec{w} \times \vec{u}$, $\vec{w} \times \vec{v}$.

Let's do this one

$\vec{w} = \langle 1, 2, 3 \rangle$

$\vec{u} = \langle 1, 1, 1 \rangle$

$\vec{w} \times \vec{u} = \langle 1, 2, 3 \rangle \times \langle 1, 1, 1 \rangle$

$= \langle 2 \cdot 1 - 3 \cdot 1, 3 \cdot 1 - 1 \cdot 1, 1 \cdot 1 - 2 \cdot 1 \rangle$

$= \langle -1, 2, -1 \rangle$.

Check:

$\langle 1,2,3 \rangle \cdot \langle -1,2,-1 \rangle = -1 + 4 - 3 = 0$ ✓

$\langle 1,1,1 \rangle \cdot \langle -1,2,-1 \rangle = -1 + 2 - 1 = 0$ ✓

Finally we get the normal equation of the plane. Take

$\langle a,b,c \rangle = \langle -1, 2, -1 \rangle$ ← we had 6 choices

$(x_0, y_0, z_0) = (4, 1, 2) \leftarrow$ we had 3 choices

Equation:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$-1(x-4) + 2(y-1) - 1(z-2) = 0$$

simplify if we want

$$-x + 2y - z + 4 - \cancel{2} + \cancel{2} = 0$$

$$-x + 2y - z = -4$$

$$1x - 2y + 1z = 4$$

That looks better. $\langle 1, -2, 1 \rangle$

Normal vector is still visible.

But the point is hidden.

(2) How can we memorize the cross product?

Most common mnemonic. Let

$$\left.\begin{aligned}\vec{i} &= \langle 1,0,0\rangle\\ \vec{j} &= \langle 0,1,0\rangle\\ \vec{k} &= \langle 0,0,1\rangle\end{aligned}\right\}\ \text{standard basis vectors}$$

so general vector is

$$\begin{aligned}\langle a,b,c\rangle &= a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle\\ &= a\vec{i} + b\vec{j} + c\vec{k}\end{aligned}$$

Mnemonic

$$\vec{u} = \langle u_1, u_2, u_3\rangle$$

$$\vec{v} = \langle v_1, v_2, v_3\rangle$$

$$\vec{u}\times\vec{v} = \text{``}\begin{matrix}\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3 & u_1 & u_2 & u_3\\ v_1 & v_2 & v_3 & v_1 & v_2 & v_3\end{matrix}\text{''}$$

$$\begin{aligned}= &+u_2v_3\vec{i} + u_3v_1\vec{j} + u_1v_2\vec{k}\\ &- u_2v_1\vec{k} - u_3v_2\vec{i} - u_1v_3\vec{j}\end{aligned}$$

Example:

$\langle 1,2,3\rangle \times \langle 1,1,1\rangle$

$$= \text{"}\begin{array}{cccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\text{"}$$

$$= +2\vec{i} + 3\vec{j} + 1\vec{k}$$
$$\quad -2\vec{k} - 3\vec{i} - 1\vec{j}$$

$$= (2-3)\vec{i} + (3-1)\vec{j} + (1-2)\vec{k}$$

$$= -1\vec{i} + 2\vec{j} - 1\vec{k}$$

$$= \langle -1, 2, -1\rangle \quad \checkmark$$

Deeper: This trick is based on the concept of a "determinant" of a square matrix, which comes from linear algebra.

2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= +aei + bfg + cdh$$

$$- ceg - bdi - afh$$

Recursive Expansion:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Laplace expansion along first row a, b, c

$$= a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix}$$

$$- b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix}$$

$$+ c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Cross product:

$$\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle$$

$$= \text{``}\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}\text{''}$$

Not literal because $\vec{i}, \vec{j}, \vec{k}$ are vectors not scalars. But whatever.

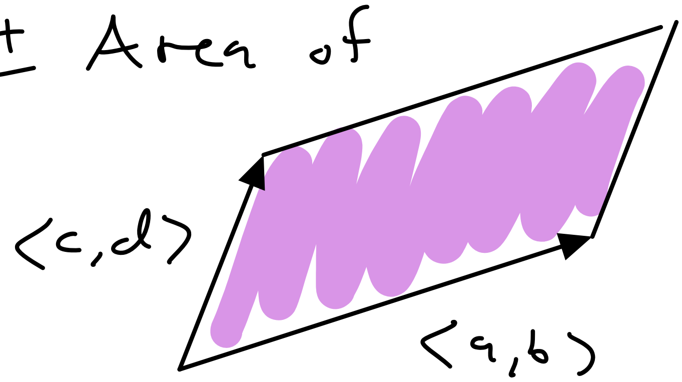
Ultimate meaning of determinants.

Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ be n vectors in n-dimensional space.

$$\underbrace{\det \begin{pmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ \vdots \\ - \vec{u}_n - \end{pmatrix}}_{n \times n \text{ matrix}} = \pm \text{ Volume of parallelepiped generated by } \vec{u}_1, \ldots, \vec{u}_n.$$

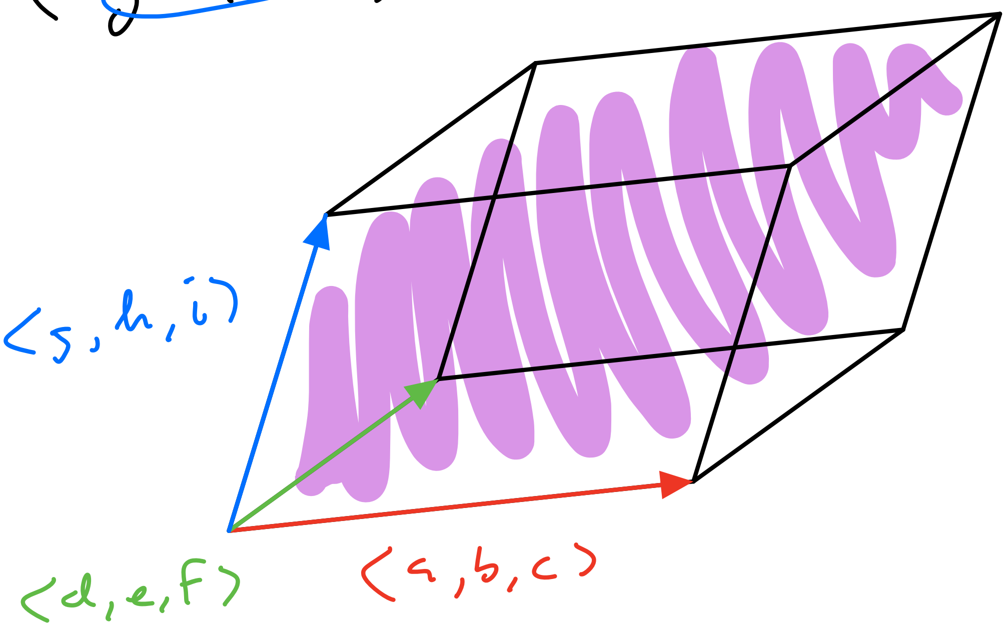
2×2 :

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \text{ Area of}$$



3×3 :

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \pm \text{ Volume of}$$



[This is behind "u-substitution" for 2D & 3D integration, as we'll see.]

Algebraic Rules for Cross Product
(Only applies in 3D)

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $a(\vec{u} \times \vec{v}) = (a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$

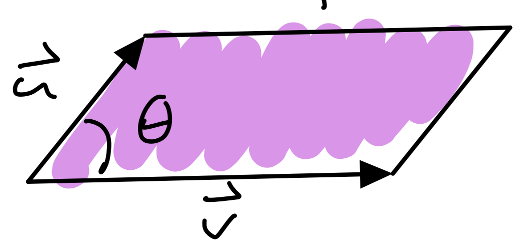
the zero vector, not the zero scalar.

- $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$
- $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$

the scalar zero.

- $\underbrace{|\vec{u} \times \vec{v}|}_{\text{length of the vector } \vec{u} \times \vec{v}} = \|\vec{u}\| \|\vec{v}\| \sin\theta$

$= \pm$ area of the parallelogram



- $\vec{u} \cdot (\vec{v} \times \vec{w}) = \pm$ volume of parallelepided spanned by $\vec{u}, \vec{v}, \vec{w}$.

Reason: $\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} - & \vec{u} & - \\ - & \vec{v} & - \\ - & \vec{w} & - \end{pmatrix}$.

Quiz 1 Discussion:

Problem 1:

$$f(t) = (x(t), y(t)) = (1+3t^2, 4t^2)$$

$$f'(t) = (0 + 3\cdot 2t, 4\cdot 2t)$$

$$= (6t, 8t)$$

$$\|f'(t)\| = \sqrt{(6t)^2 + (8t)^2}$$

$$= \sqrt{36t^2 + 64t^2}$$

$$= \sqrt{100t^2}$$

$$= 10t$$

Arc length $t = 0 .. 1$

$$= \int_0^1 10t \, dt$$

$$= 10\left[\frac{t^2}{2}\right]_0^1 = 5.$$

What does the curve look like?

Eliminate t:

$x = 1 + 3t^2 \longrightarrow t^2 = (x-1)/3$

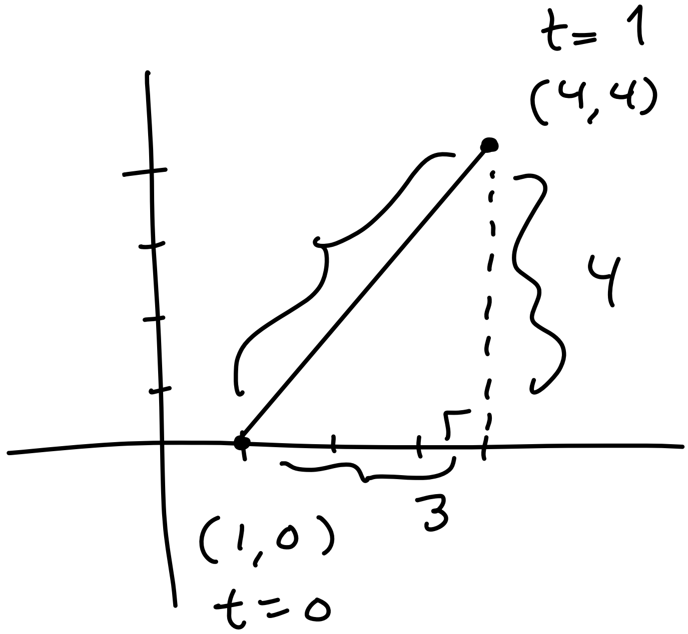
$y = 4t^2 \longrightarrow t^2 = y/4$

$$\frac{(x-1)}{3} = \frac{y}{4}$$

$$4(x-1) = 3y.$$

$$4x - 3y = 4 \qquad \text{Line!}$$

Picture:



$$\text{Arc Length}^2 = 3^2 + 4^2 = 25$$

$$\text{Arc Length} = 5 \quad \checkmark$$

Moving on with Chapters 2 & 3.

Sketch:

Chap 2 & 3: Vectors, Vector-valued functions (i.e., parametrized paths).
Motion in space & integration of vector-valued functions.
Lines & Planes.

Chapter 4: Differentiation in any # of dimensions...
In particular, GRADIENTS.

Chapter 5: Integration in any # of dimensions.
e.g. surface area, volume, physics (total work/energy, ...)

Chapter 6: Putting it all together.
Div, Grad, Curl stuff.

Function $f : \mathbb{R} \to \mathbb{R}^2$

or $\mathbb{R} \to \mathbb{R}^3$

Think of as a parametrized curve in $\mathbb{R}^2$ or $\mathbb{R}^3$. New notation:

$$\vec{f} : \mathbb{R} \to \mathbb{R}^3$$

$$\vec{r} : \mathbb{R} \to \mathbb{R}^3$$

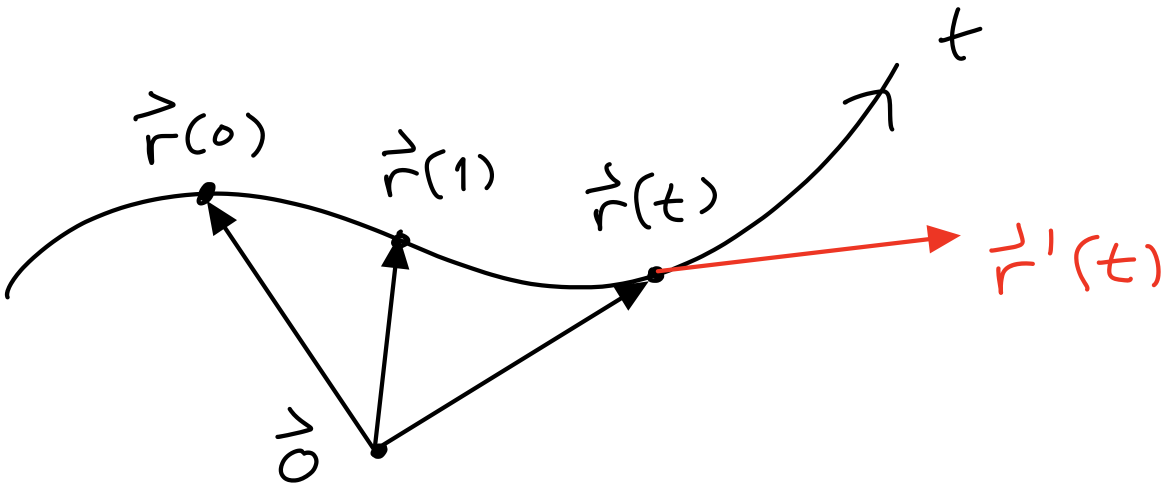
the output is a vector.

Notation

$$\vec{r}(t) = (x(t), y(t), z(t))$$

is common in physics. [I think $\vec{r}$ stands for "radius".]

Picture:



Parametrized line in any # of dimensions:

$$\vec{r}(t) = \vec{x}_0 + t\,\vec{v}$$

initial position (at time 0)

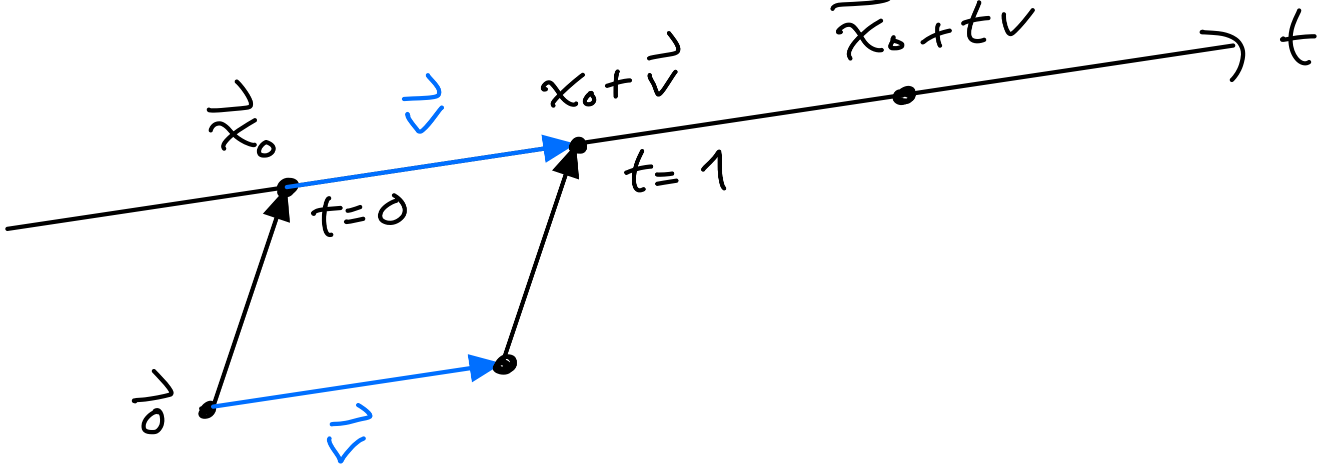
velocity.

e.g. in 3D.

$$\vec{x}_0 = (x_0, y_0, z_0)$$

$$\vec{v} = \langle a, b, c \rangle$$

$$\vec{r}(t) = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$$



We know the equation of a line in $\mathbb{R}^2$ & a plane in $\mathbb{R}^3$:

$$a(x-x_0)+b(y-y_0) = 0$$

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

What is the equation of a <u>line in $\mathbb{R}^3$</u>?

TRICK QUESTION!

A line in $\mathbb{R}^3$ cannot be described with only one equation. We need at least 2 equations.

e.g. Consider a parametrized line

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}.$$

Try to eliminate t:

$$t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This gives us 3 different equations involving x, y, z (but not t):

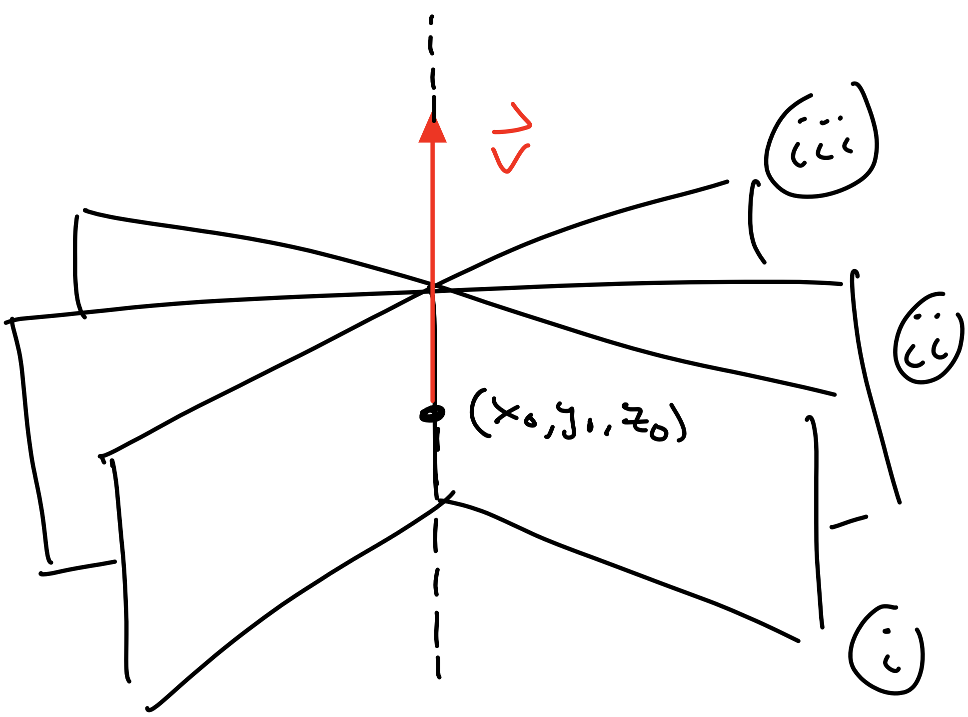
(i) $(x-x_0)/a = (y-y_0)/b$

(ii) $(x-x_0)/a = (z-z_0)/c$

(iii) $(y-y_0)/b = (z-z_0)/c$.

Each of these represents a plane

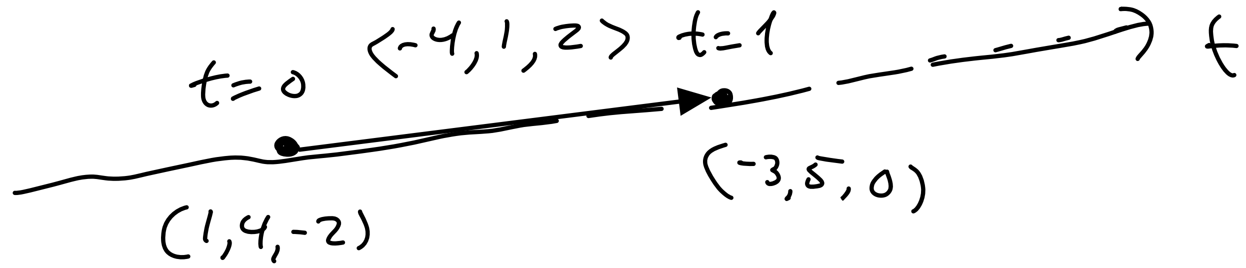
Any two of these planes intersect at the original line:



i, ii, iii are called the "symmetric equations" of the line. But there are ∞ many pairs of equations that describe this line.

Example: Find a parametrization & symmetric equations for the line in $\mathbb{R}^3$ through points

$$P = (1, 4, -2) \ \& \ Q = (-3, 5, 0).$$



Initial point $\vec{x}_0 = (1, 4, -2)$

velocity $\vec{v} = \langle -4, 1, 2 \rangle$

Parametrization:

$$\vec{r}(t) = (1-4t, 4+t, -2+2t).$$

$$\text{OR} \quad \begin{cases} x = 1-4t \\ y = 4+t \\ z = -2+2t \end{cases}$$

Eliminate t to obtain the symmetric equations:

$$t = \frac{x-1}{-4} = \frac{y-4}{1} = \frac{z+2}{2}$$

So our line is at the intersection of the following 3 planes:

(i) $(x-1)/(-4) = y-4$

$(x-1) = -4y + 16$

$x + 4y = 17$

(ii) $(x-1)/(-4) = (z+2)/2$

$2(x-1) = (-4)(z+2)$

$2x - 2 = -4z - 8$

$2x + 4z = -6$

$x + 2z = -3$

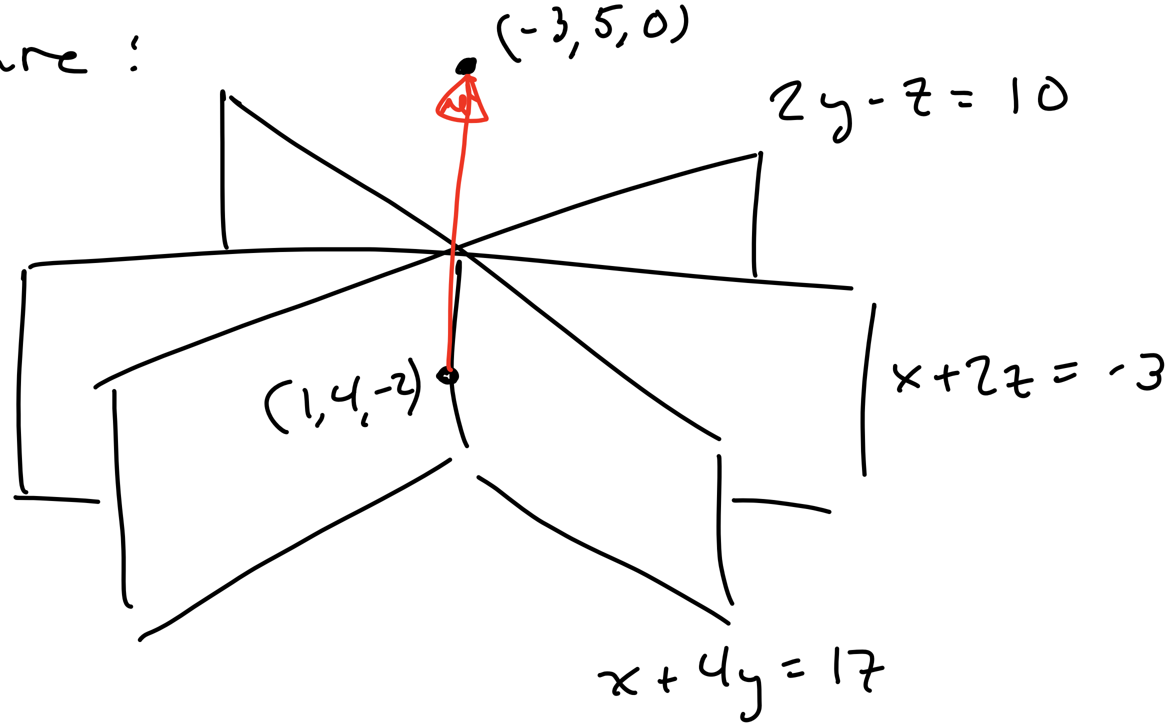
(iii) $(y-4)/1 = (z+2)/2$

$2(y-4) = z+2$

$2y - 8 = z + 2$

$2y - z = 10$

Picture:



Conversely, suppose we are given two planes. Find a parametrization for the line of intersection.

$$\begin{array}{l} ① \\ ② \end{array} \begin{cases} ax + by + cz = d \\ Ax + By + Cz = D \end{cases}$$

$$\rightsquigarrow \vec{r}(t) = (x_0 + tu,\ y_0 + tv,\ z_0 + tw)$$

Example:

$$\begin{array}{l} ① \\ ② \end{array} \begin{cases} x + y + z = 4, \\ x + 2y + 3z = 3. \end{cases}$$

We'll use the method of "elimination".

First subtract equations to eliminate x:

$$
\begin{array}{rl}
 & (\cancel{x} + 2y + 3z = 3) \\
- & (\cancel{x} + y + z = 4) \\
\hline
3 & \quad y + 2z = -1
\end{array}
$$

Get new equation 3 with no x.

This gives a simpler, but equivalent, system of equations:

$$
\begin{array}{l}
1 \\
3
\end{array}
\begin{cases}
x + y + z = 4, \\
\quad\; y + 2z = -1.
\end{cases}
$$

Finally, we use eq 3 to eliminate y from eq 1. Take 1 − 3

$$
\begin{array}{rl}
 & (x + \cancel{y} + z = 4) \\
- & (0 + \cancel{y} + 2z = -1) \\
\hline
\end{array}
$$

$$④\quad x + 0 - z = 5$$

Our final equivalent system is

$$\begin{cases} ④ & \boxed{x + 0} - z = 5, \\ ③ & \boxed{0 + y} + 2z = -1. \end{cases}$$

The good thing: We have "solved" for the "pivot variables" x & y, in terms of the "free variable" z.

Let's write down the solution

$$\begin{cases} x = 5 + z \\ y = -1 - 2z. \end{cases}$$

This looks like a parametrized line with parameter z.

WEIRD: Let's define $t = z$.

Then we really do get a parametrized line:

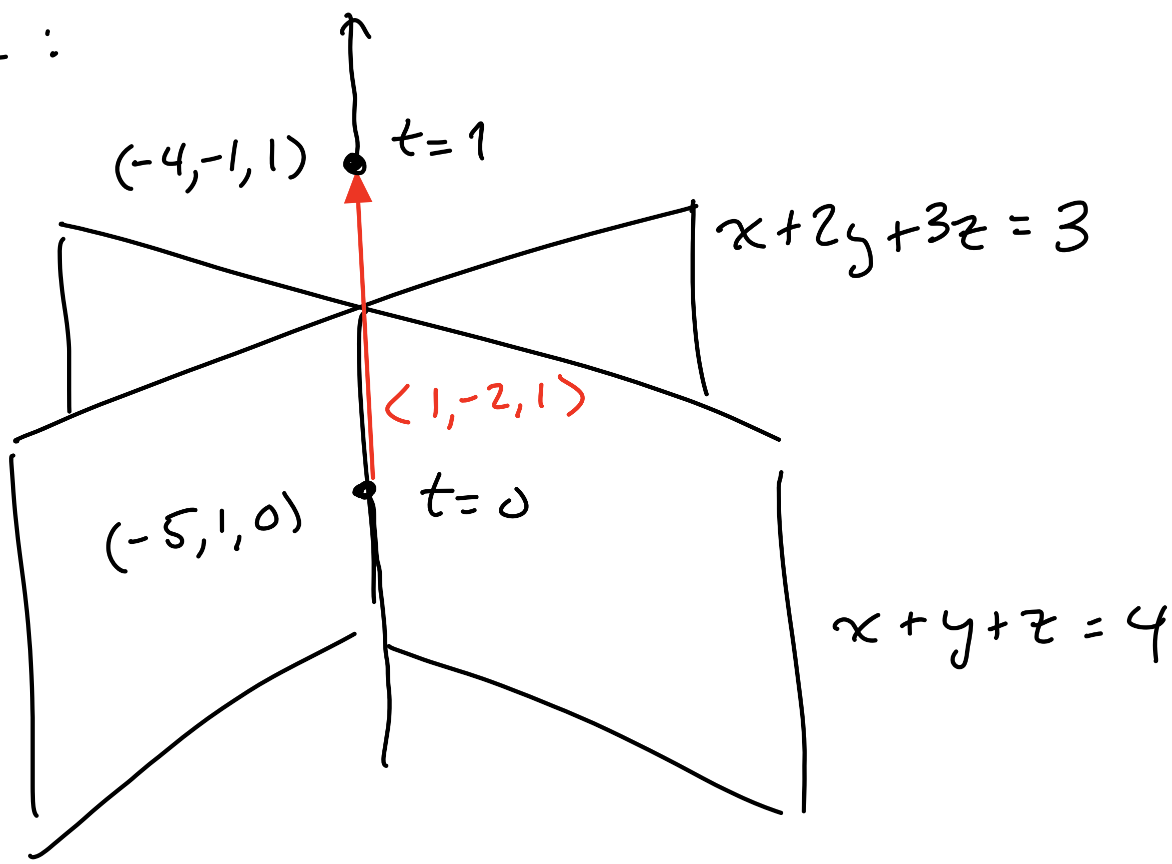
$$\begin{cases} x = 5 + t \\ y = -1 - 2t \\ z = t \end{cases}$$ ✓

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$= (5 + t, -1 - 2t, 0 + t)$$

$$(5, -1, 0) + t\,(1, -2, 1)$$

Picture:



There is no such thing as "the equation of a line" in $\mathbb{R}^3$.

Instead we need at least 2 equations.

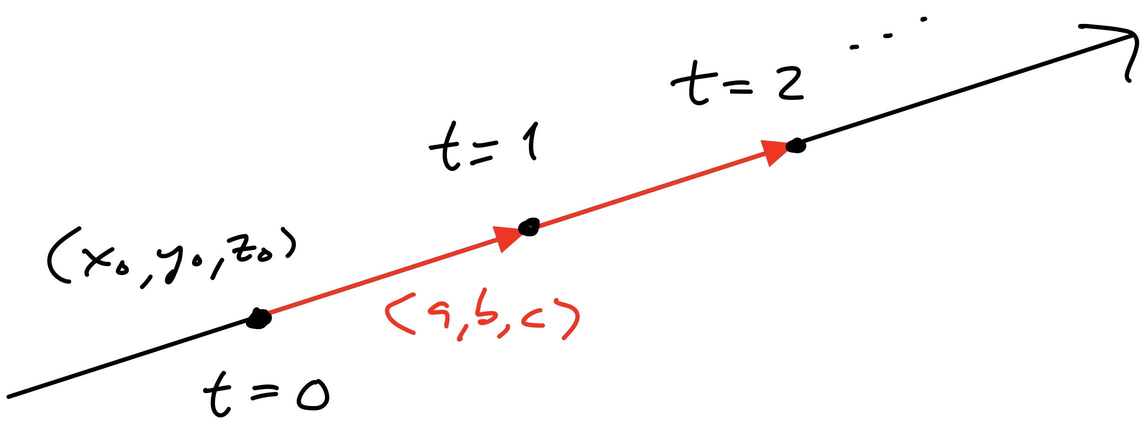
Geometrically: A line is an intersection of 2 planes in $\mathbb{R}^3$.

Two ways to describe a line in $\mathbb{R}^3$.

- Parametrization:

$$\vec{r}(t) = \vec{x}_0 + t\vec{a}$$

$$= \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$$



One parameter because a line is a "one dimensional" shape.

- As the solution of a system of 2 linear equations in 3 unknowns.

Example:

$$\begin{cases} x - y + 0 = 1, & (1) \\ x + y + 2z = 1. & (2) \end{cases}$$

To find a parametrization of the line we let $t = z$ be the parameter, then solve for x & y in terms of t. Method:

Find an equation without x and an equation without y.

$$(1): \quad x - y = 1$$

$$(2): \quad x + y = 1 - 2\overset{t}{\cancel{z}}$$

$$(1) + (2): \quad 2x + 0 = 2 - 2\overset{t}{\cancel{z}}$$

$$(2) - (1): \quad 0 + 2y = 0 - 2\overset{t}{\cancel{z}}$$

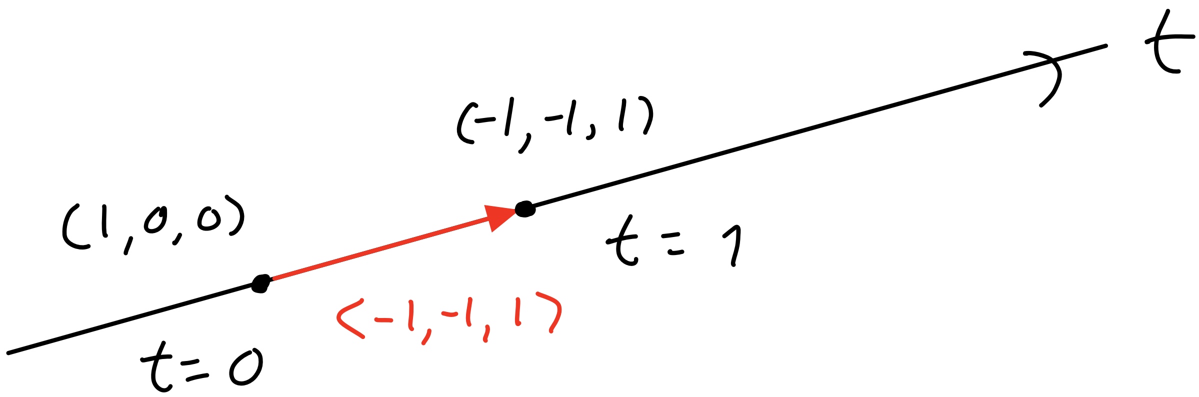
We conclude that

$$\begin{cases} x = 1 - t \\ y = -t \\ z = t \end{cases}$$

$$\vec{r}(t) = \langle x, y, z \rangle = \langle 1-t, -t, t \rangle$$

$$= \langle 1-t, 0-t, 0+t \rangle = \langle 1,0,0 \rangle + \langle -t, -t, t \rangle$$

$$= \langle 1,0,0 \rangle + t \langle -1,-1,1 \rangle$$



We can also give a "parametric description" of a plane in $\mathbb{R}^3$.

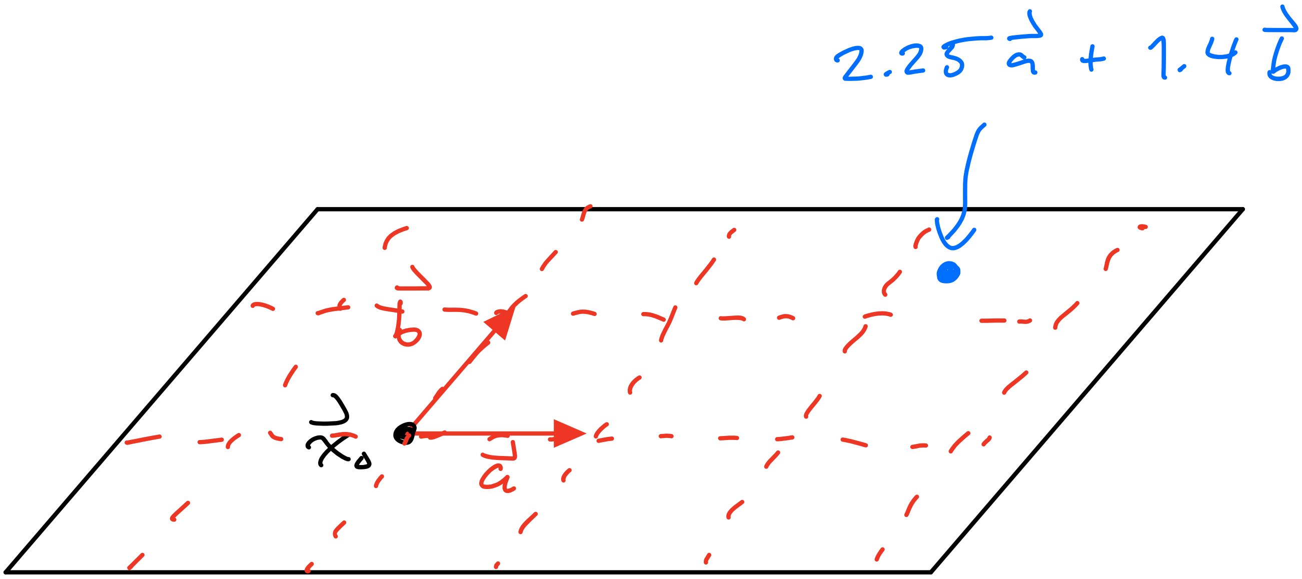
Since a plane is "2 dimensional" we will need 2 independent parameters.

What should we call them?

Our book uses u & v.

A parametric plane has the form

$$\vec{r}(u,v) = \vec{x}_0 + u\vec{a} + v\vec{b}$$

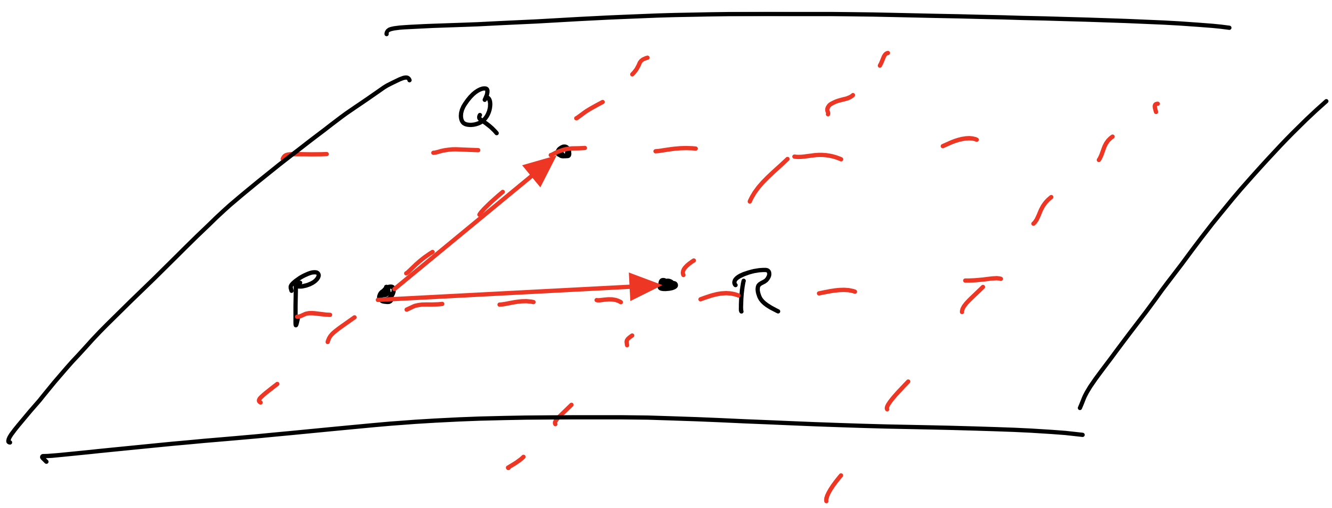


Every point in the plane has unique "coordinates" u, v with respect to the "basic vectors" $\vec{a}$ & $\vec{b}$. This is the only way to introduce a "coordinate system" on a general plane: Pick a point & pick a "basis" of two direction vectors.

Example: The plane generated by points $P = (1,1,0)$

$Q = (1,0,2)$

$R = (1,2,3)$



$$\vec{r}(u,v) = P + u\,\vec{PQ} + v\,\vec{PR}$$

$$= \langle 1,1,0\rangle + u\langle 0,-1,2\rangle + v\langle 0,2,1\rangle$$

$$= \langle 1,\ 1-u+2v,\ 0+2u+v\rangle$$

Example: Parametrize the plane

$$2x + 3y - z = 5$$

- Find 3 points ?

- Easiest: Let $y=u$ & $z=v$ be parameters. Then solve for x.

$$2x + 3u - v = 5$$
$$2x = 5 - 3u + v$$
$$x = \frac{5}{2} - \frac{3}{2}u + \frac{1}{2}v$$
$$y = u$$
$$z = v$$

$$\vec{r}(u,v) = \left\langle \overbrace{\frac{5}{2} - \frac{3}{2}u + \frac{1}{2}v}^{x},\ \overset{y}{u},\ \overset{z}{v} \right\rangle$$
$$= \left\langle \frac{5}{2} - \frac{3}{2}u + \frac{1}{2}v,\ 0 + 1u + 0v,\ 0 + 0u + 1v \right\rangle$$
$$= \left\langle \frac{5}{2}, 0, 0 \right\rangle + u\left\langle -\frac{3}{2}, 1, 0 \right\rangle + v\left\langle \frac{1}{2}, 0, 1 \right\rangle.$$

Later we will parametrize general surfaces in $\mathbb{R}^3$. We need to do this so we can <u>integrate</u> over a surface.

Example: Let $\vec{r}(u,v)$ be a parametrized surface. Let

$$\vec{r}_u(u,v) = \frac{d}{du}\vec{r}(u,v)$$

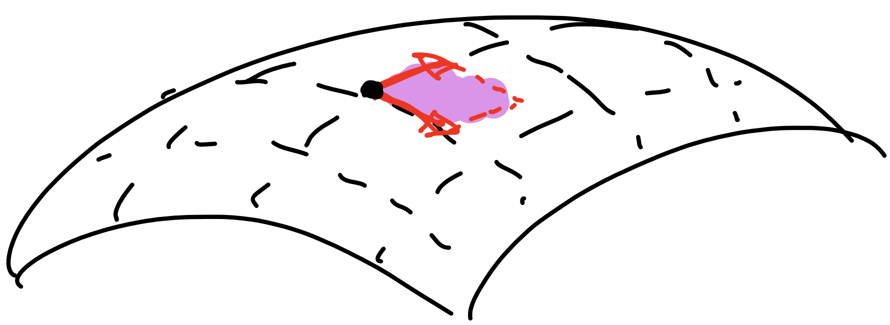
$$\vec{r}_v(u,v) = \frac{d}{dv}\vec{r}(u,v)$$

The <u>surface area</u> is a double integral

$$\iint |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

integrate over all relevant values of u, v.

Reason: $|\vec{r}_u \times \vec{r}_v|\, du\, dv$ is the area of a tiny parallelogram.



Stay Tuned.

Motion in Space & Newton's Second Law.

Consider a parametrized curve in 3D:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Think of t as time.

At each time there is a velocity vector & an acceleration vector:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle.$$

Example:

$$\vec{r}(t) = (x_0 + at, y_0 + bt, z_0 + ct)$$

$$\vec{v}(t) = \langle a, b, c \rangle \quad \text{CONSTANT.}$$

$$\vec{a}(t) = \langle 0, 0, 0 \rangle. \quad \text{NO ACCELERATION.}$$

Conversely, let $\vec{r}(t)$ be any path with zero acceleration:

$$\vec{r}''(t) = \langle 0, 0, 0 \rangle.$$

Integrate to find the velocity.

$$\vec{r}'(t) = \left\langle \int 0\,dt, \int 0\,dt, \int 0\,dt \right\rangle$$

$$= \langle a, b, c \rangle$$

for some constants of integration a, b, c. Then integrate velocity to get position:

$$\vec{r}(t) = \left\langle \int a\,dt, \int b\,dt, \int c\,dt \right\rangle$$

$$= \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

for some constants of integration; call them x_0, y_0, z_0.

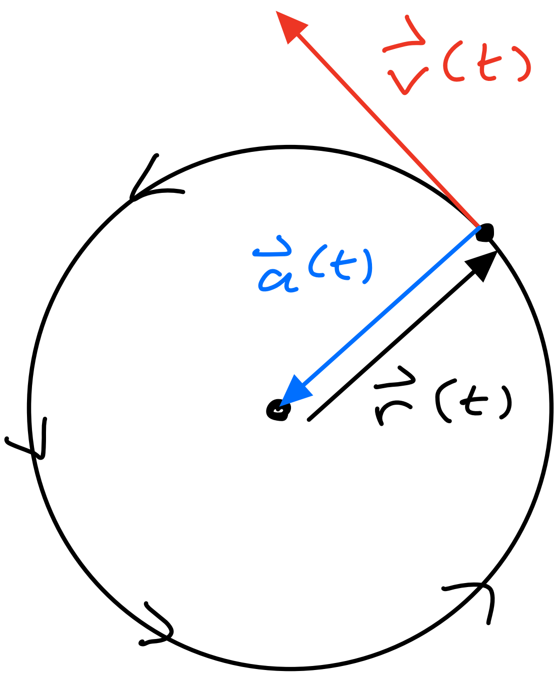
Conclusion: Any path with zero acceleration is a <u>straight line</u> with constant velocity.

Circle : $\vec{r}(t) = (\cos t, \sin t)$

$\vec{v}(t) = \langle -\sin t, \cos t \rangle$

$\vec{a}(t) = \langle -\cos t, -\sin t \rangle$.

Picture :

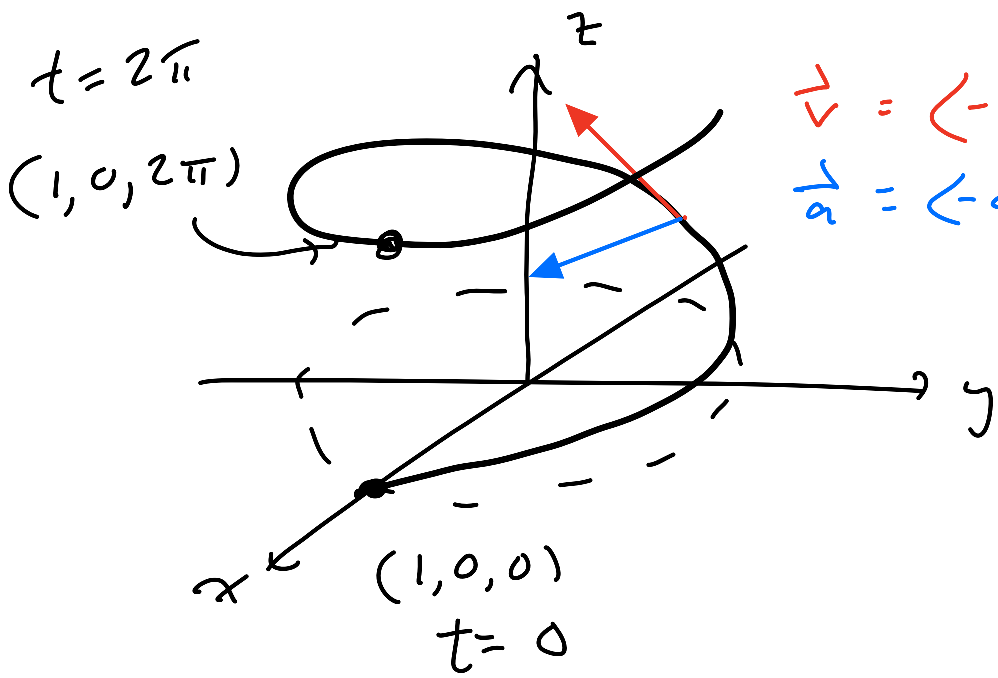


Acceleration always points directly to the center because.

$$\vec{a}(t) = \langle -\cos t, -\sin t \rangle = -\vec{r}(t).$$

Example in $\mathbb{R}^3$: Consider the following "helical" path

$$\vec{r}(t) = (x(t), y(t), z(t))$$
$$= (\cos t, \sin t, t)$$



$\vec{v} = \langle -\sin t, \cos t, 1 \rangle$

$\vec{a} = \langle -\cos t, -\sin t, 0 \rangle$

Find arc length between $t = 0$ & $t = 2\pi$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{v}(t)\| = \sqrt{\underbrace{\cancel{\sin^2 t + \cos^2 t}}_{1} + 1}$$

$$= \sqrt{2}.$$

$$\text{Arc length} = \int_0^{2\pi} \|\vec{v}(t)\| \, dt$$

$$= \int_0^{2\pi} \sqrt{2} \, dt$$

$$= 2\pi\sqrt{2}.$$

Acceleration vector:

$$\vec{a}(t) = \vec{v}\,'(t) = \langle -\sin t, \cos t, 1 \rangle'$$

$$= \langle -\cos t, -\sin t, 0 \rangle.$$

Points directly toward z-axis.

Integration of vector-valued functions.

Given any function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

we define the integral with respect to t "componentwise"

$$\int_{t_0}^{t_1} \vec{r}(t)\,dt =$$

$$\left\langle \int_{t_0}^{t_1} x(t)\,dt,\ \int_{t_0}^{t_1} y(t)\,dt,\ \int_{t_0}^{t_1} z(t)\,dt \right\rangle$$

Why do we want to do this?

Newton's 2nd Law.

If a time varying force $\vec{F}(t)$ acts on a particle of mass m at point $\vec{r}(t)$, Then we have

force = mass · acceleration

$$\vec{F}(t) = m\,\vec{a}(t)$$
$$= m\,\vec{v}\,'(t)$$
$$= m r''(t).$$

This is a second order differential equation. Goal: Solve for $\vec{r}(t)$.

Example: Projectile Motion.

Near the surface of the Earth, a free falling body experiences a constant acceleration due to gravity:

$$\vec{r}''(t) = \langle 0, 0, -9.81 \text{ m/s}^2 \rangle$$

We can solve for $\vec{r}(t)$ by integrating twice.

$$\vec{r}'(t) = \left\langle \int 0\,dt, \int 0\,dt, \int -9.81\,dt \right\rangle$$

$$= \langle c_1, c_2, -9.81t + c_3 \rangle$$

Meaning of the constants:

$$\vec{r}'(0) = \langle c_1, c_2, 0 + c_3 \rangle$$

initial velocity.
Call it u_0, v_0, w_0

$$\vec{r}'(t) = \langle u_0, v_0, -9.81t + w_0 \rangle.$$

Integrate again to get position at time t:

$$\vec{r}(t) = \left\langle \int \quad dt, \int \quad dt, \int (-9.81t + \quad)\,dt \right\rangle$$

$$= \left\langle u_0 t + d_1, \; v_0 t + d_2, \; -\frac{9.81}{2}t^2 + w_0 t + d_3 \right\rangle$$

Meaning of d_1, d_2, d_3:

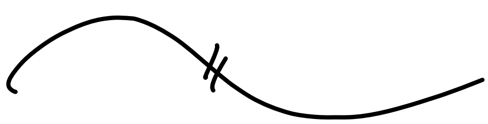
$$\vec{r}(0) = \langle \underbrace{d_1, d_2, d_3}_{\text{initial position.}} \rangle$$

Call x_0, y_0, z_0

A free falling body has position

$$\vec{r}(t) = \langle \underbrace{x_0 + u_0 t, \; y_0 + v_0 t, \; z_0 + w_0 t}_{\text{this part is like a straight line}} - \frac{9.81}{2}t^2 \rangle$$

this makes it curve toward Earth.



More interesting:

Universal Gravitation.

Put the sun at $(0,0,0)$ in $\mathbb{R}^3$.

A moving planet has position $\vec{r}(t)$.

Newton: Planet feels a gravitational force of magnitude

$$\frac{GMm}{\|\vec{r}(t)\|^2}$$

where M = mass of sun
m = mass of planet
G = gravitational constant.

The direction of the force is directly toward the sun.

SOLVE FOR $\vec{r}(t)$.

HW 2 due Friday before class.

Motion in Space.

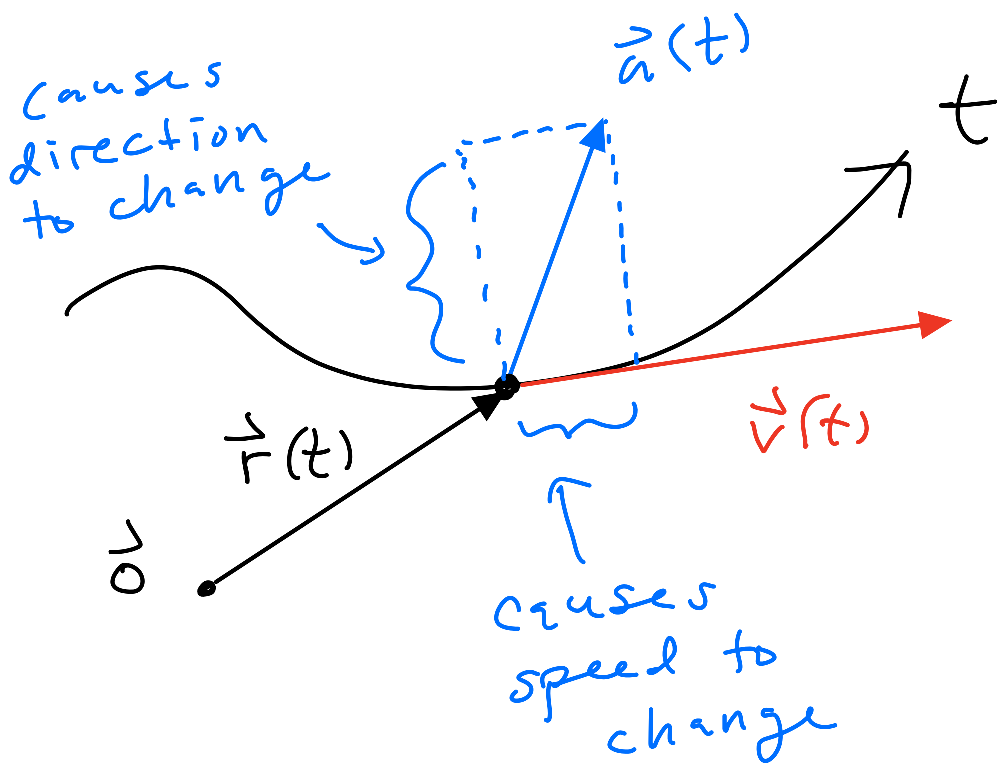
Given parametrized curve in $\mathbb{R}^3$:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\vec{a}(t) = \vec{v}\,'(t)$$
$$= \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle.$$

Picture:



Newton's Second Law:

force = mass · acceleration.

If a force $\vec{F}(t)$ acts on a particle will mass m & position $\vec{r}(t)$ then we have

$$\vec{F}(t) = m\,\vec{r}\,''(t)$$

Example: Gravity.

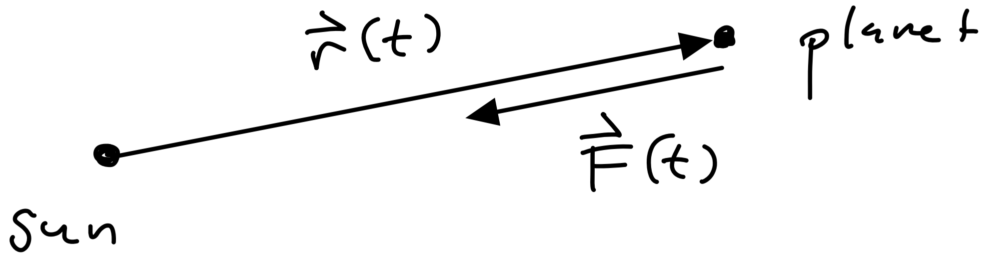
The sun is at origin $(0,0,0)$ in $\mathbb{R}^3$.

A planet has position $\vec{r}(t)$.

Let $\vec{F}(t)$ be the gravitational force felt by the planet. Then:

- $\vec{F}(t)$ points directly toward the sun.
- $\|\vec{F}(t)\| = GMm / \|\vec{r}(t)\|^2$

where G = gravitational constant
M = mass of sun
m = mass of planet.



$\vec{F}(t) = \ ?$

Know: $\quad \vec{F}(t) = -c(t)\,\vec{r}(t)$

for some scalar $c(t)$.

Use the fact that

$$\|c\vec{v}\| = |c|\,\|\vec{v}\|.$$

[<u>Proof</u>: $\|c\vec{v}\|^2 = (c\vec{v})\cdot(c\vec{v})$

$$= c^2\,\vec{v}\cdot\vec{v}$$

$$= c^2\,\|\vec{v}\|^2$$

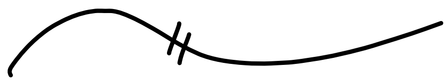
$$\|c\vec{v}\| = \sqrt{c^2\|\vec{v}\|^2}$$

$$= |c|\,\|\vec{v}\|.$$]

Know $\|\vec{F}(t)\| = GMm/\|\vec{r}(t)\|^2$.

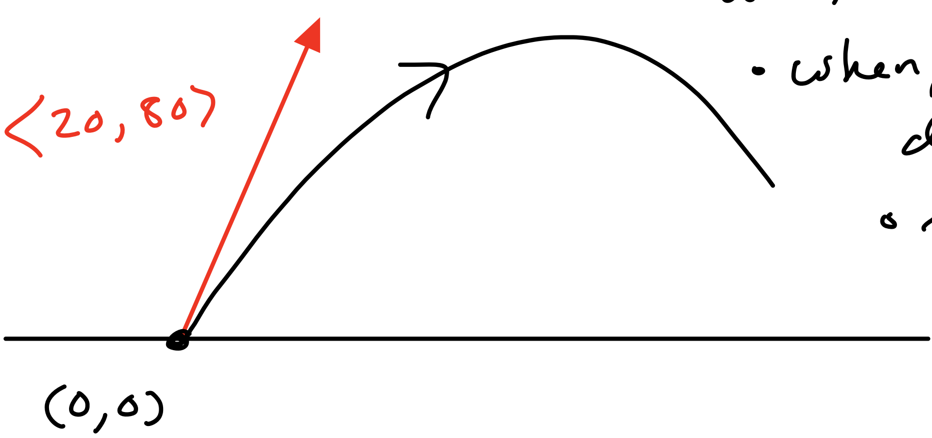
And $\|\vec{F}(t)\| = |c(t)|\,\|\vec{r}(t)\|$

So $c(t) = ?$ HW 2 Problem 5.



Easier : Projectile Motion near surface of the Earth.

Projectile will travel in a 2D plane, so we'll just describe in the x,y- plane.



- what shape?
- when/where does it land?
- how high?

Galileo: $\vec{r}''(t)$ is <u>constant</u>.

$\vec{r}\,''(t) = \langle 0, -32 \text{ feet/sec}^2 \rangle$

[This is a "textbook problem" so the numbers will be nice.]

Integrate to get velocity.

$\vec{v}(t) = \vec{r}\,'(t) = \int \vec{r}\,''(t)\,dt.$

$= \left\langle \int 0\,dt, \int -32\,dt \right\rangle$

$= \langle c_1, -32t + c_2 \rangle$

To find constants, sub $t = 0$:

$\vec{v}(0) = \langle c_1, -32(0) + c_2 \rangle = \langle c_1, c_2 \rangle$

GIVEN $\vec{v}(0) = \left\langle 20 \frac{\text{feet}}{\text{sec}}, 80 \frac{\text{feet}}{\text{sec}} \right\rangle.$

So $c_1, c_2 = 20, 80.$

$\vec{v}(t) = \langle 20, 80 - 32t \rangle.$

Integrate again to get position:

$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \vec{r}\,'(t)\,dt$

$$= \left\langle \int 20\,dt, \int (80-32t)\,dt \right\rangle$$

$$= \left\langle 20t + c_3, 80t - 16t^2 + c_4 \right\rangle$$

To find c_3, c_4, sub $t=0$.

$$\vec{r}(0) = \langle c_3, c_4 \rangle$$

GIVEN $\vec{r}(0) = \langle 0, 0 \rangle$.

So $c_3 = 0$ & $c_4 = 0$ hence

$$\boxed{\vec{r}(t) = \langle 20t, 80t - 16t^2 \rangle}$$

Shape? Eliminate t:

$$x = 20t \quad \rightsquigarrow \quad t = x/20.$$

$$y = 80t - 16t^2$$

$$y = 80\left(\frac{x}{20}\right) - 16\left(\frac{x}{20}\right)^2$$

$$y = 4x - \frac{1}{25}x^2$$

It's a parabola!

Where/When does it land?

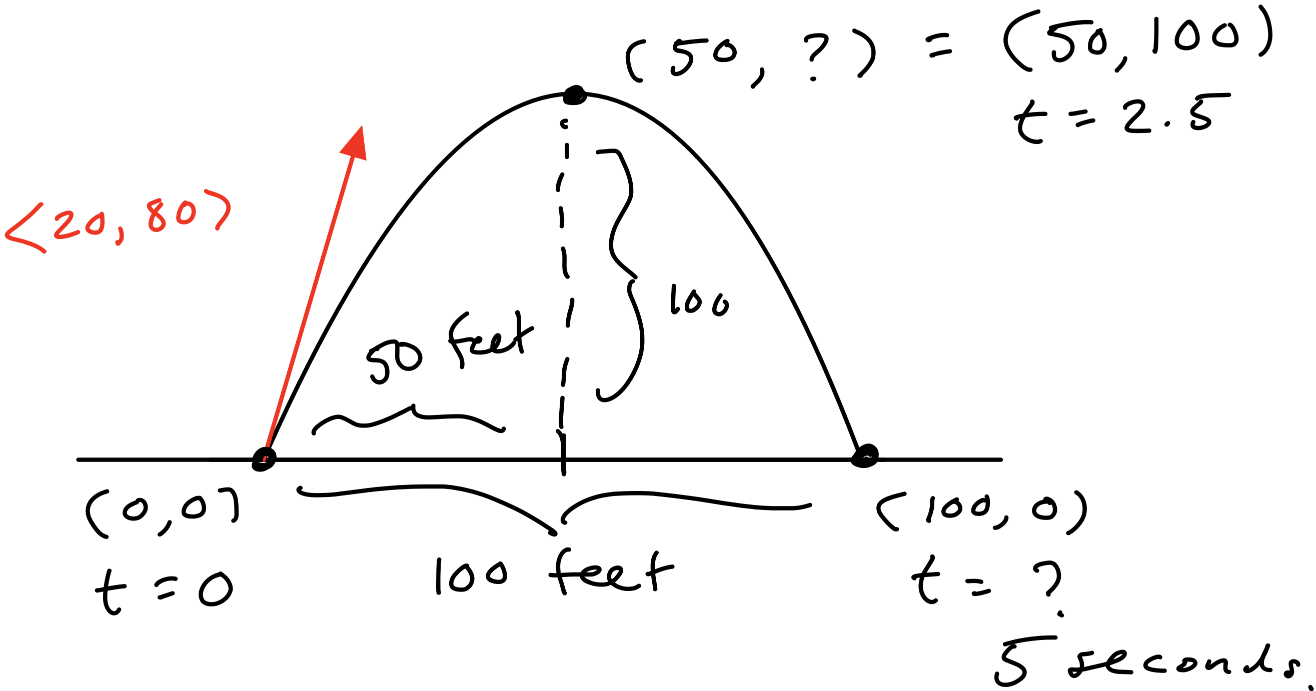
$y = 0.$

$4x - \frac{1}{25}x^2 = 0$

$100x - x^2 = 0$

$x(100 - x) = 0$

$\rightsquigarrow \quad x = 0 \quad \text{or} \quad x = 100.$



when?

$y(t) = 0$

$80t - 16t^2 = 0$

$t(80 - 16t) = 0$

$t = 0 \quad \text{or} \quad 80 - 16t = 0$

$t = 80/16 = 5$ seconds.

How High ?

Two ways :

- Use $y = 4x - \frac{1}{25}x^2$

Parabolas are symmetric.

slope at top $= 0$

$dy/dx = 0$

$4 - \frac{2}{25}x = 0$

$x = \frac{4 \cdot 25}{2} = 50$ ✓

so $y = 4(50) - \frac{1}{25}(50)^2$

$= 200 - 2 \cdot 50 = 100$ feet.

- Use $\vec{r}(t) = \langle 20t, 80t - 16t^2 \rangle$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{80 - 32t}{20} = 0.$ Horizontal Tangent.

$80 - 32t = 0$

$t = 80/32 = 2.5$ seconds.

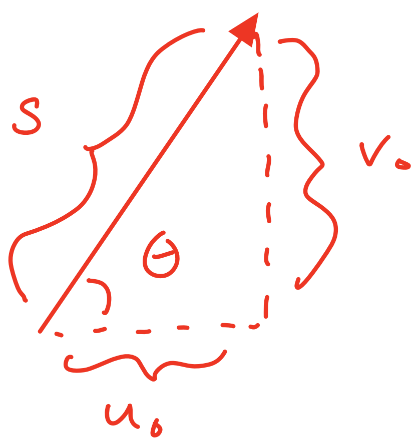
More generally:

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$
$$\vec{v}(0) = \langle u_0, v_0 \rangle$$
$$\vec{a}(t) = \langle 0, -g \rangle$$

Then integrating twice gives

$$\vec{r}(t) = \langle x_0 + u_0 t,\ y_0 + v_0 t - \tfrac{1}{2} g t^2 \rangle$$

More common to describe $\vec{v}(0)$ in terms of speed and angle.



$$u_0 = s\cos\theta$$
$$v_0 = s\sin\theta.$$

$$\vec{r}(t) = \langle x_0 + s\cos\theta \cdot t,\ y_0 + s\sin\theta \cdot t - \tfrac{1}{2} g t^2 \rangle$$

See HW 2 Problem 3.

Maximize the horizontal distance traveled.

Differentiation Rules:

Consider some vector-valued functions:

$$\vec{u} : \mathbb{R} \to \mathbb{R}^n$$
$$\vec{v} : \mathbb{R} \to \mathbb{R}^n$$

$$\vec{u}(t) = \langle u_1(t), u_2(t), \dots, u_n(t) \rangle$$
$$\vec{v}(t) = \langle v_1(t), v_2(t), \dots, v_n(t) \rangle$$

Also consider scalar $c \in \mathbb{R}$ and a regular function

$$f : \mathbb{R} \to \mathbb{R}.$$

Then we have the following rules:

$$[c\,\vec{u}(t)]' = c\,\vec{u}'(t)$$

$$[\vec{u}(t) \pm \vec{v}(t)]' = \vec{u}'(t) \pm \vec{v}'(t)$$

$$\Big[\overbrace{f(t)\,\vec{u}(t)}^{\text{vector}}\Big]' = \underbrace{f'(t)\,\vec{u}(t)}_{\text{vector}} + \underbrace{f(t)\,\vec{u}'(t)}_{\text{vector}}.$$

scalar that changes

vector that changes

$$\Big[\underbrace{\vec{u}(t)\cdot\vec{v}(t)}_{\text{scalar that changes}}\Big]' = \underbrace{\vec{u}'(t)\cdot\vec{v}(t)}_{\text{scalar that changes}} + \underbrace{\vec{u}(t)\cdot\vec{v}'(t)}_{\text{scalar that changes.}}.$$

$$\Big[\underbrace{\vec{u}(f(t))}_{\text{vector}}\Big]' = \underbrace{\vec{u}'(f(t))}_{\text{vector}}\cdot\underbrace{f'(t)}_{\text{scalar}}$$

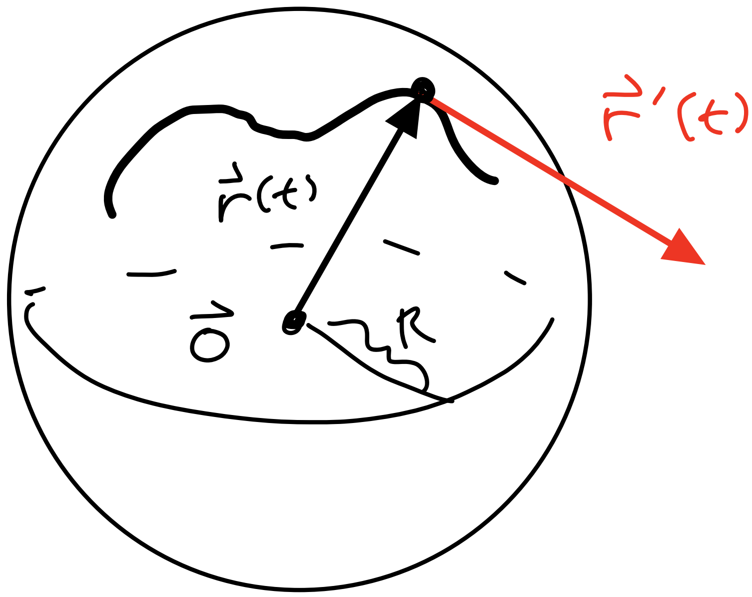
Just like Calc I :)

If we are working in $\mathbb{R}^3$ then we also have a "product rule" for cross products:

$$\Big[\underbrace{\vec{u}(t)\times\vec{v}(t)}_{\text{vector}}\Big]' = \underbrace{u'(t)\times\vec{v}(t)}_{\text{vector}} + \underbrace{\vec{u}(t)\times\vec{v}'(t)}_{\text{vector}}$$

Good news: Easy to memorize.

Application: Suppose particle travels on surface of a sphere of radius R.



I claim that $\vec{r}(t) \perp \vec{r}'(t)$ for all times t. (The velocity is always tangent to sphere.)

Proof: GIVEN

$$\|\vec{r}(t)\| = R \quad \text{for all } t.$$

$$\|\vec{r}(t)\|^2 = R^2$$

$$\underbrace{\vec{r}(t) \cdot \vec{r}(t)}_{\text{scalar}} = \underbrace{R^2}_{\text{scalar}}$$

Differentiate both sides with resp. to t.

$$\left[\vec{r}(t) \cdot \vec{r}(t)\right]' = 0$$

$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0 \quad \checkmark$$

JUST ALGEBRA !

[Recall : $2(\vec{u} \cdot \vec{v}) = (2\vec{u}) \cdot \vec{v} = \vec{u} \cdot (2\vec{v})$]

Preview of next Topic:

How should we think of a function

$$f : \mathbb{R}^n \to \mathbb{R}.$$

For any vector $\vec{v}$ in $\mathbb{R}^2$

we get a scalar $f(\vec{v}) \in \mathbb{R}$.

OR: For any point $P = (x_1, x_2, \ldots, x_n)$

we get a scalar

$$f(x_1, x_2, \ldots, x_n) \in \mathbb{R}.$$

This could represent

- temperature at a point
- pressure
- density
- chemical concentration

⋮

etc.

We are attaching a number to each point in space.

Called a SCALAR FIELD.

How can we visualize this?

Example: The temperature at the point (x, y) in $\mathbb{R}^2$ is

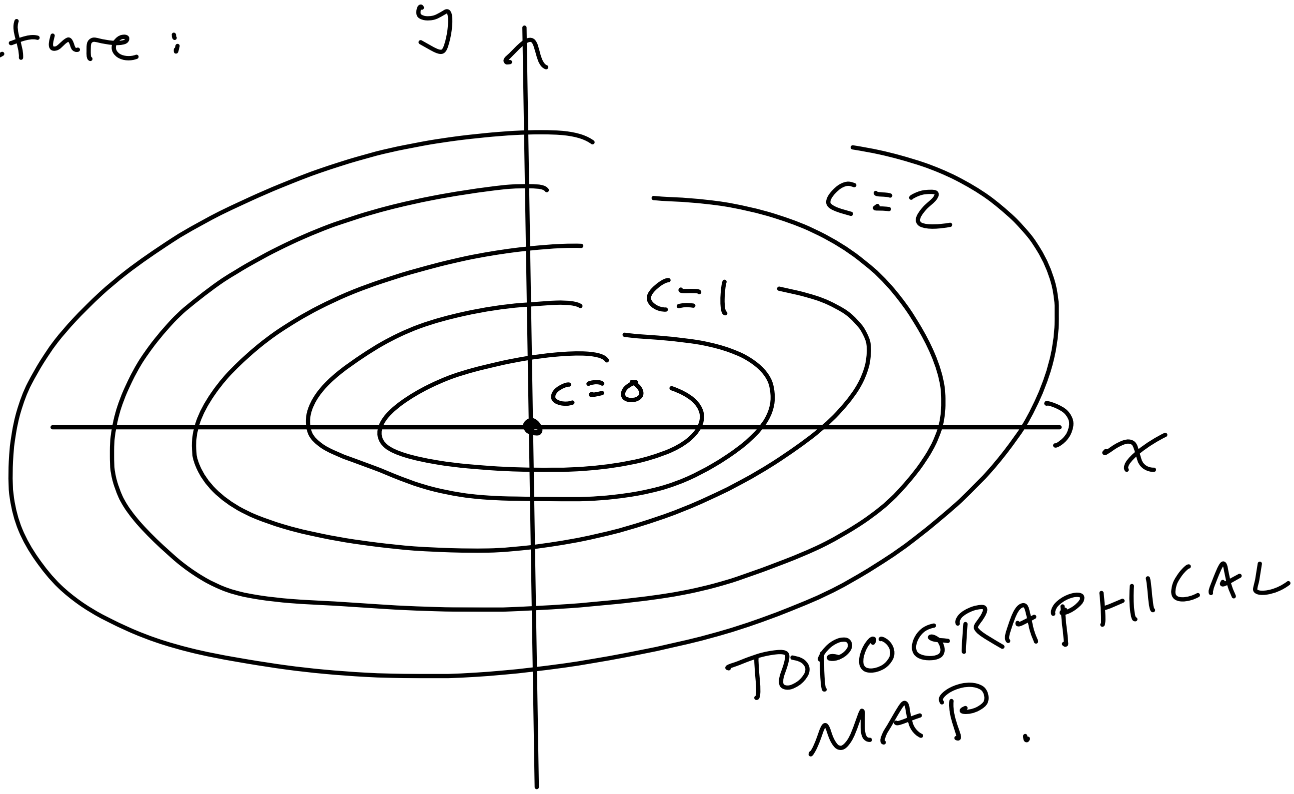
$$f(x,y) = \left(\frac{x}{2}\right)^2 + y^2.$$

For each fixed temperature c, the set of points with this temperature is an ellipse

$f(x,y) = c$

$\left(\frac{x}{2}\right)^2 + y^2 = c.$

Picture:



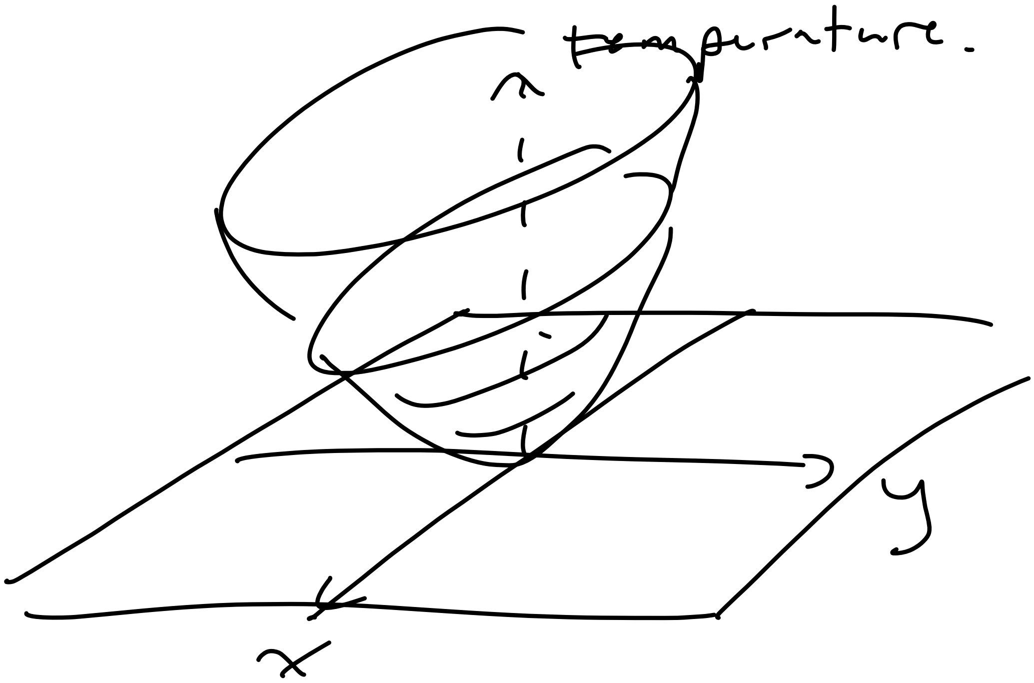
These ellipses called "isotherms" or "curves of constant temperature".

Also call them the "level curves" of the function $f(x,y)$.

Better: Think of temperature as a "third variable".

Then view $f(x,y)$ as a "2D

surface in $\mathbb{R}^3$"



The surface is a parabolic bowl.
This surface is just the "graph"
of the function $F(x,y)$.