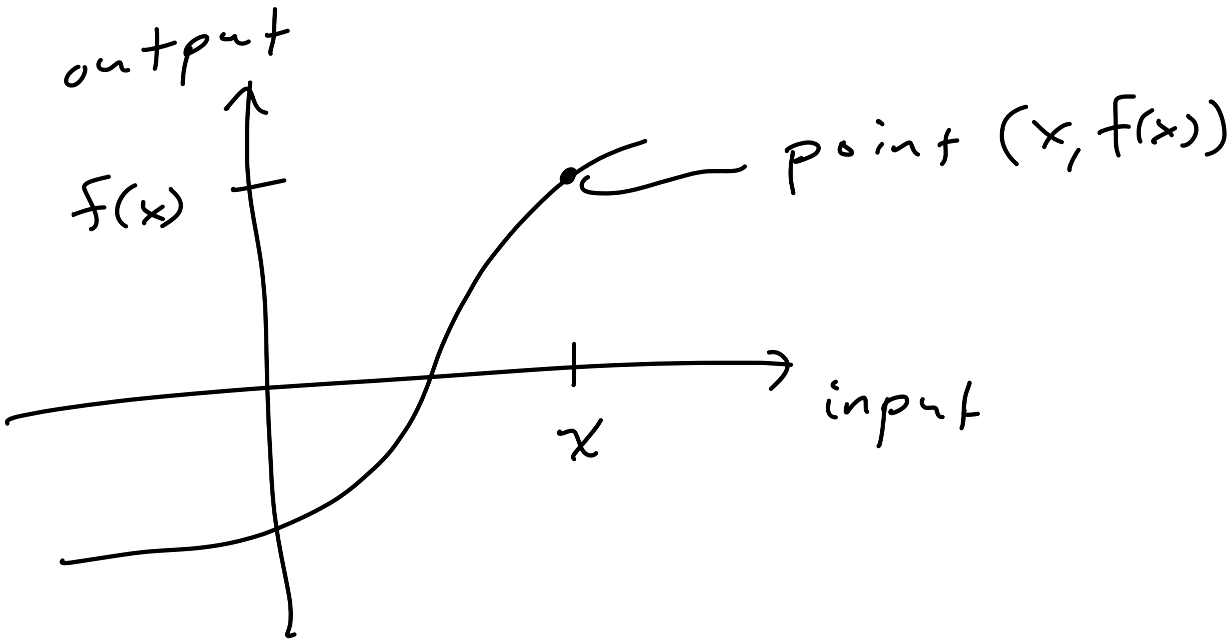
M211 : Calculus 3.

Calc 1 & 2 are based on functions

$$f : \mathbb{R} \to \mathbb{R}$$

one real input → one real output.

Such functions are visualized by considering their "graph":



Calc 3 is based on functions

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

m real inputs → n real outputs

In particular for $m, n = 1, 2, 3$.

This is more relevant to the real world (physics) because the real world is 3D.

Such functions are harder to visualize. We will develop some intuitions:

- $f : \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$ is a parametrized path in the plane or space.

- $f : \mathbb{R}^2$ or $\mathbb{R}^3 \longrightarrow \mathbb{R}$ is called a "scalar field". It associates a number (e.g. temperature) to each point in the plane or space.

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$ are called "vector fields",

e.g. electric field or
gravitational field, . . .

This week: Chapter 2 with a brief look at Chapter 1.

Consider a function $f: \mathbb{R} \to \mathbb{R}^2$.
We will use the notation

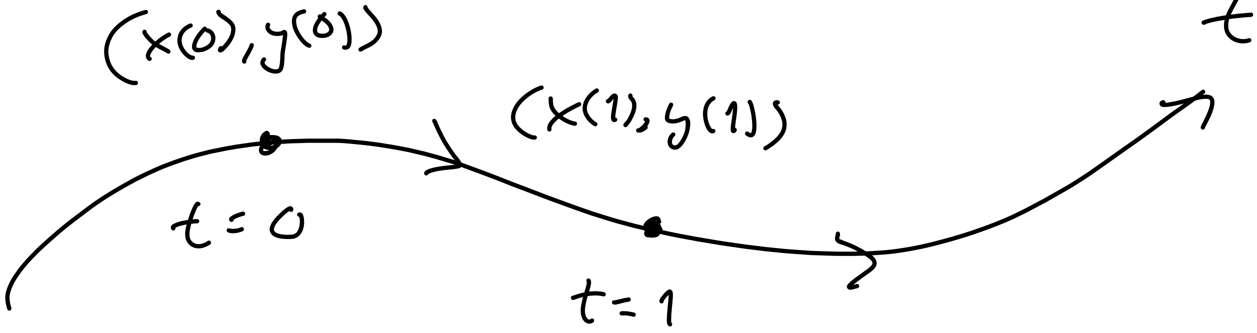
$$f(t) = (x(t), y(t))$$

input called t for "time"

outputs $x(t), y(t)$ are functions of t.

Think: $(x(t), y(t))$ is the position of a moving particle in the real x,y-plane.

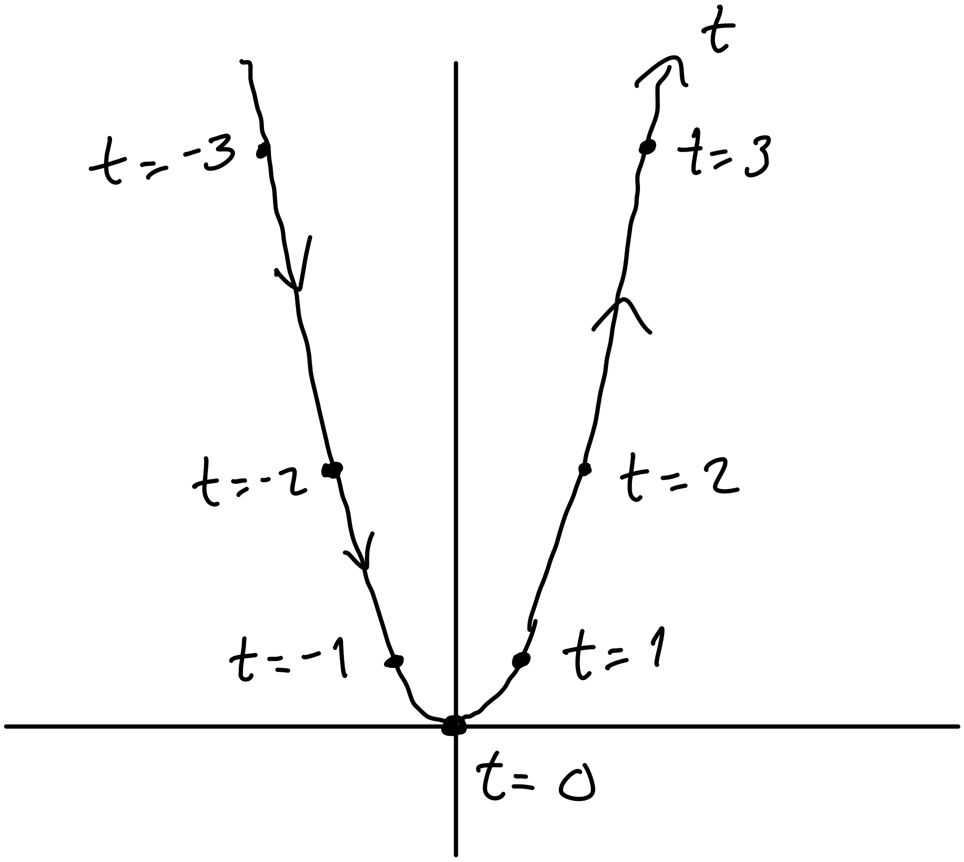
Picture:



Examples: $f(t) = (t, t^2)$.

i.e. let $x(t) = t$ & $y(t) = t^2$.

What does it look like?



It looks like a parabola.
Actually it is a parabola. We can see this by "eliminating t":

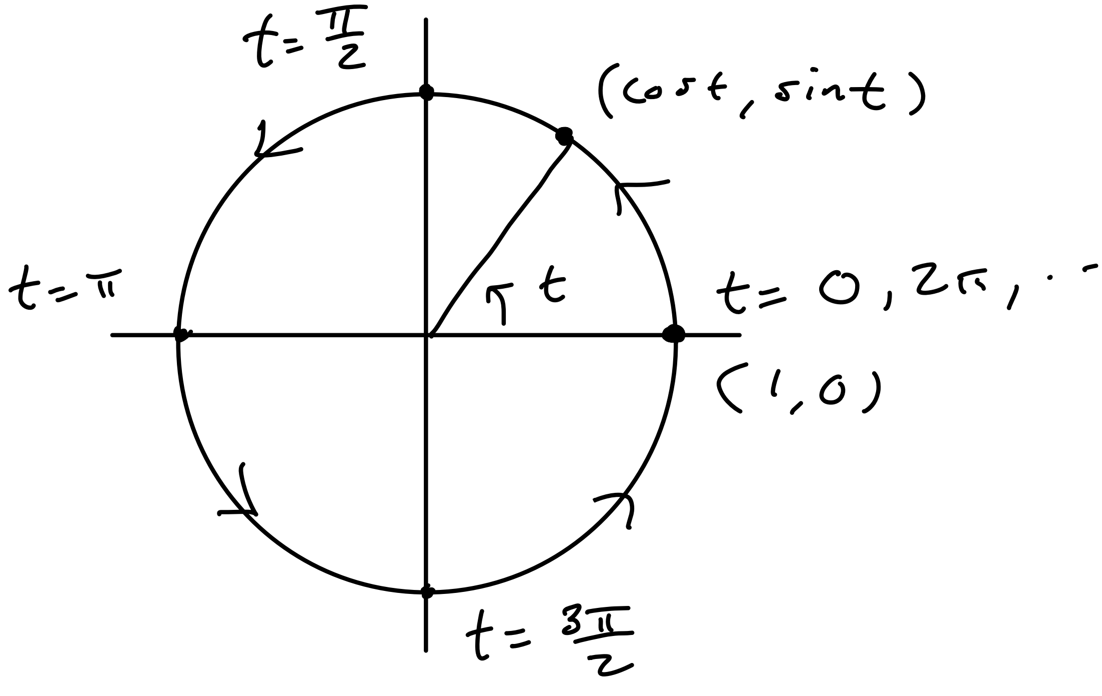
$$x = t \rightsquigarrow x^2 = t^2$$
$$y = t^2$$

so $y = x^2$ (parabola).

- Let $g(t) = (\cos t, \sin t)$

i.e. $x(t) = \cos t$
$y(t) = \sin t$.

What does it look like?



Path travels the unit circle counter-clockwise, repeats every 2π units of time.

Get the equation of the circle by "eliminating t":

$$x = \cos t$$
$$y = \sin t$$

TRICK!

$$x^2+y^2 = \cos^2 t + \sin^2 t = 1.$$

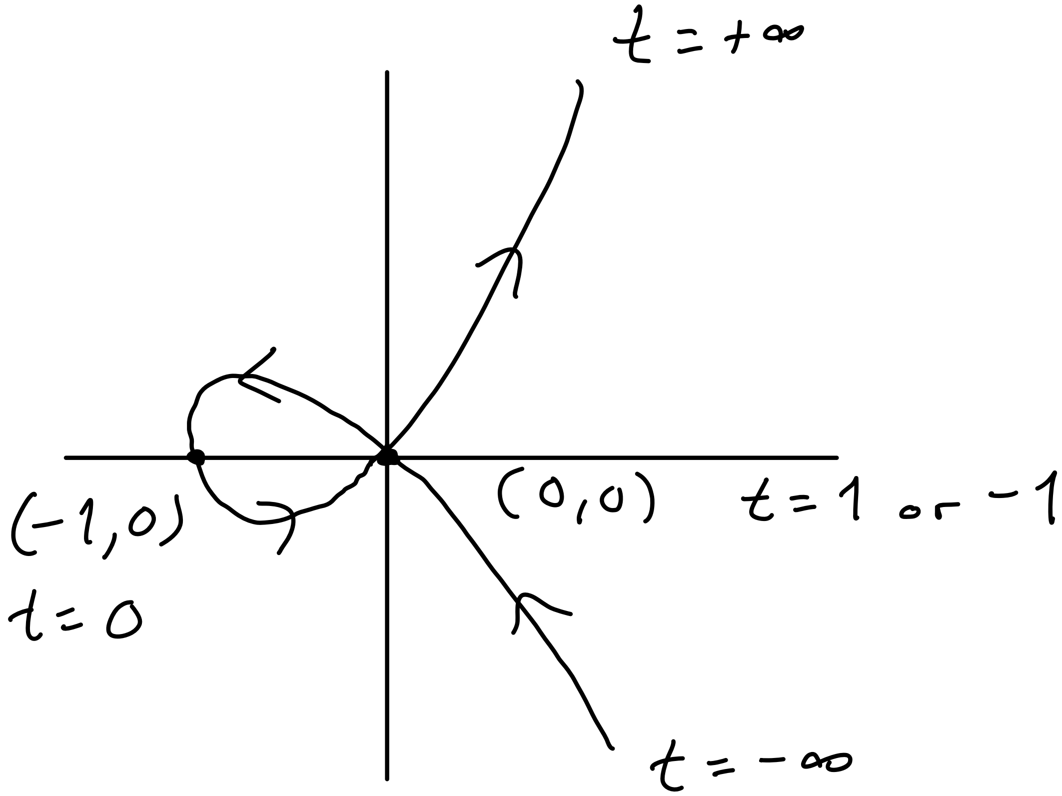
$$x^2 + y^2 = 1$$

(unit circle)

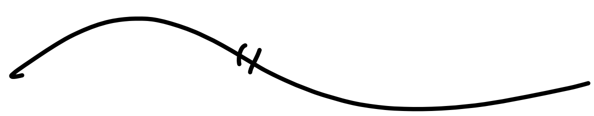
- $h(t) = (t^2-1, t^3-t)$.

What does this look like?

Plot some points:



Interesting: This path intersects itself.



Velocity & Speed.

Given function $f : \mathbb{R} \to \mathbb{R}^2$ written as $f(t) = (x(t), y(t))$ we define its derivative (with respect to t) as

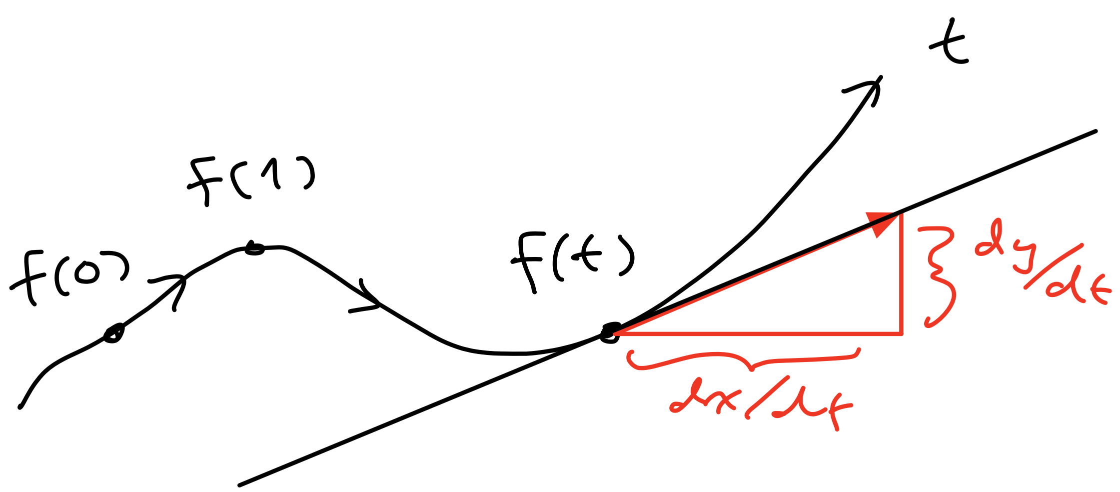
$$f' : \mathbb{R} \to \mathbb{R}^2$$

$$f'(t) = (x'(t), y'(t))$$

$$\frac{df}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

Call this the "instantaneous velocity of the parametrized path $f(t)$ at time t.

New Idea: Velocity is a <u>vector</u>.

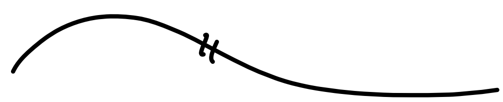


Picture: Velocity is <u>tangent</u> to the path. Hence the slope of the tangent line is

$$\frac{\text{rise}}{\text{run}} = \frac{dy/dt}{dx/dt} = \text{“}\frac{dy}{dx}\text{”}$$

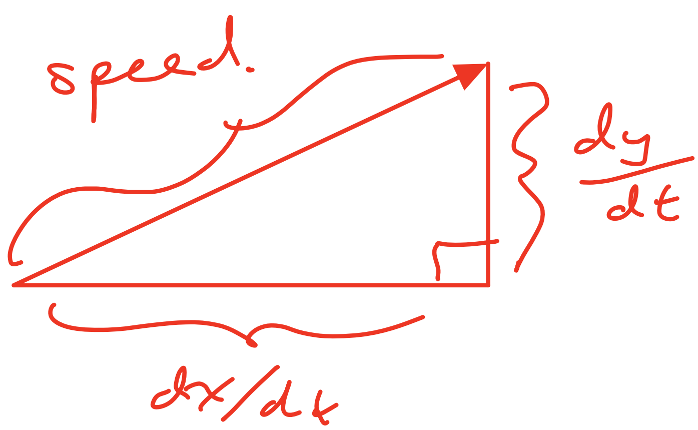
To repeat: Think of $f : \mathbb{R} \to \mathbb{R}^2$ as a parametrized path in $\mathbb{R}^2$.

Think of derivative $f' : \mathbb{R} \to \mathbb{R}^2$ as the velocity vectors of the path.



velocity is a vector.

Speed is the length or magnitude of the velocity vector:



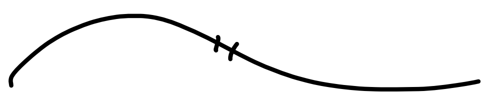
Pythagorean theorem:

$$\text{speed}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= x'(t)^2 + y'(t)^2$$

$$\text{speed} = \sqrt{x'(t)^2 + y'(t)^2}$$

= "instantaneous speed at time t"



Recall : Suppose your car has speed $s(t)$ at time t. How far do you travel ?

Between times $t = a$ & b your car travels distance

$$\text{distance} = \int_a^b s(t)\,dt$$

[speed = distance / time

$$s(t) = \frac{d}{dt}\text{distance}.$$

$$\cancel{\int \frac{d}{dt}}(\text{distance}) = \int s(t)\,dt$$

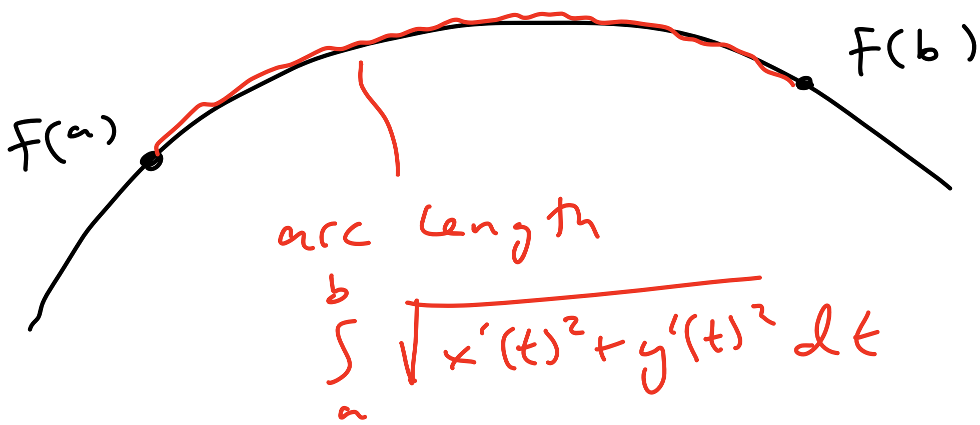
$$\text{distance} = \int \text{speed}$$

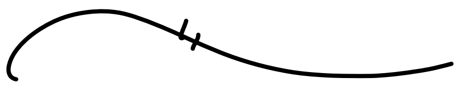
The same formula holds in higher dimensions. Given path $f(t) = (x(t), y(t))$, the <u>distance</u> (or "arc length") between $t = a$ & b is

$$\text{distance} = \int_a^b \text{speed}\,dt$$

$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Picture:





Examples:

- Parametrized Parabola

$$f(t) = (t, t^2)$$

$$f'(t) = (1, 2t) \quad \text{velocity.}$$

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1^2 + (2t)^2}$$

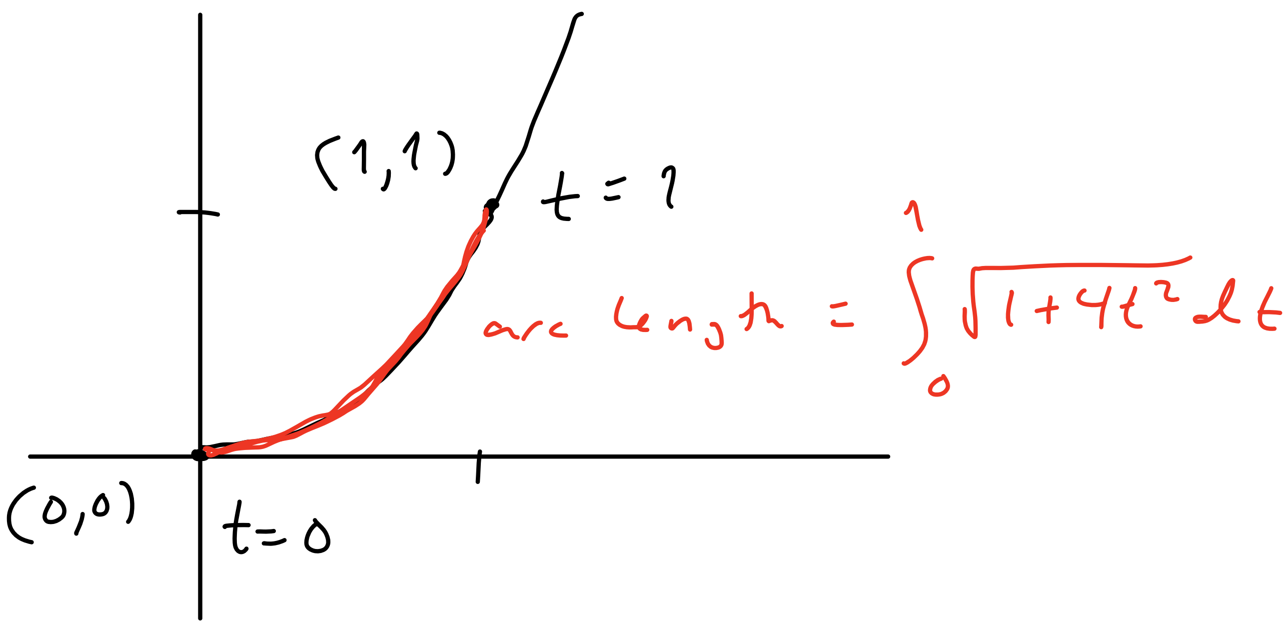
$$= \sqrt{1 + 4t^2} \quad \text{speed.}$$

So the arc length between

times $t=a$ & $t=b$ is

$$\int_a^b \sqrt{1+4t^2}\,dt$$

Say $a=0$ & $b=1$.



Do you know how to compute this?

No. Me neither.

Computer: $\displaystyle\int_0^1 \sqrt{1+4t^2}\,dt \approx 1.479$

[Sadly, most arc length integrals cannot be solved by hand!]

- parametrized unit circle:

$$f(t) = (\cos t, \sin t)$$

$$f'(t) = (-\sin t, \cos t)$$

$$\text{speed} = \sqrt{(-\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= \sqrt{1} = 1$$

The speed is constant.

So the arc length is easy to compute.

e.g. circumference

$$= \text{arc length between } 0 \ \& \ 2\pi$$

$$= \int_0^{2\pi} 1 \, dt = \Big[t \Big]_0^{2\pi} = 2\pi - 0$$

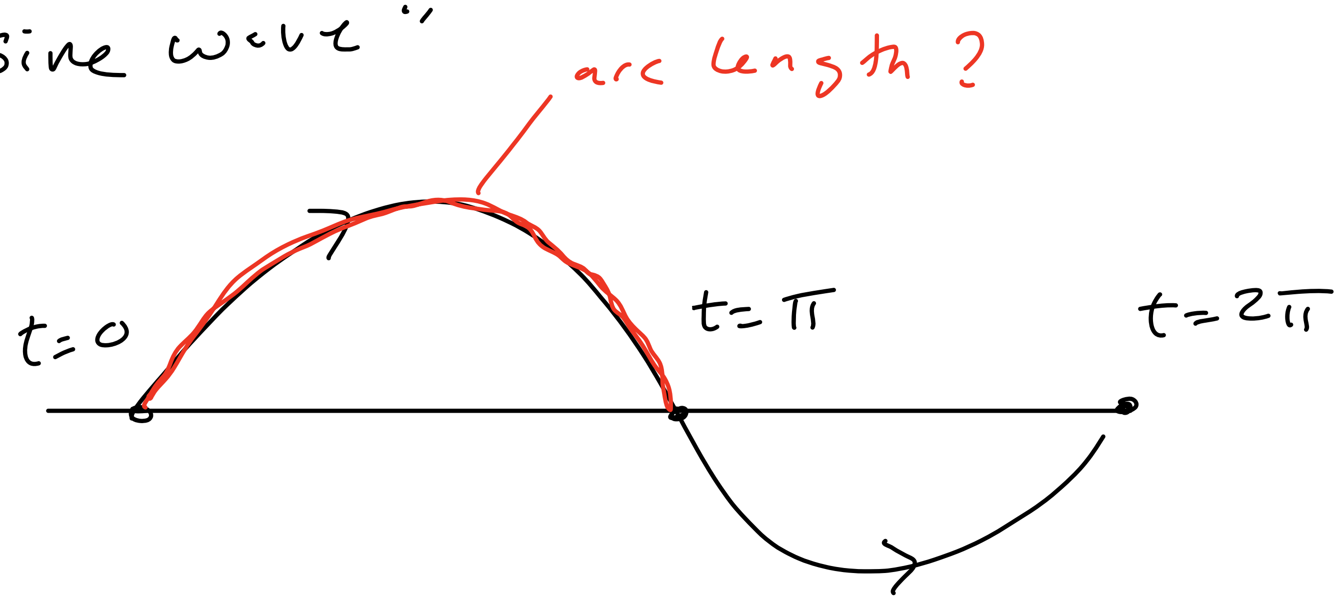
$= 2\pi$.

Yes, this is the circumference of a circle with radius 1. ✓

[HW 1: Do the same thing for a circle of radius r.]

- consider curve $f(t) = (t, \sin t)$.

"sine wave"

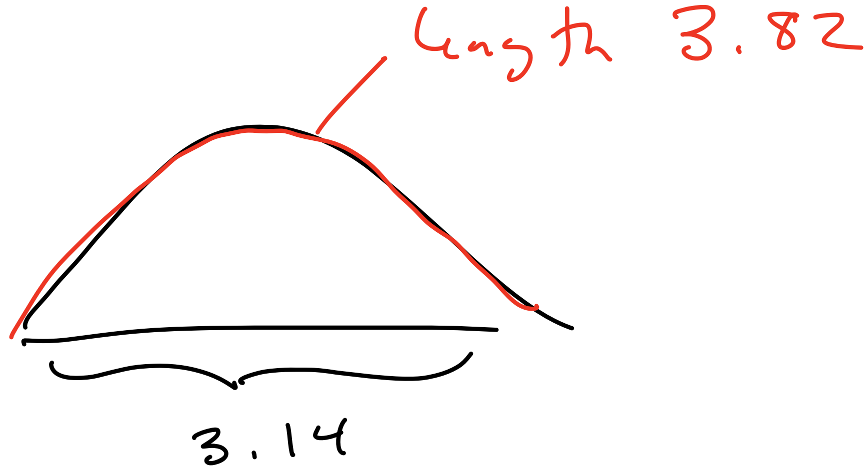


velocity $f'(t) = (1, \cos t)$

speed $\sqrt{1^2 + \cos^2 t}$

$$\text{Arc length} = \int_0^{\pi} \sqrt{1+\cos^2 t}\, dt$$

$$\approx 3.82 \quad \text{(via computer)}$$



HW 1 will be posted today, due Fri.

Recall:

$\mathbb{R}$ = the set of real numbers

= the number line.

$\mathbb{R}^2$ = ordered pairs of real numbers

= the coordinate plane.

⋮

$\mathbb{R}^n$ = ordered n-tuples of real numbers

= coordinate "n-space"

A function $f : \mathbb{R} \to \mathbb{R}^2$ can be thought of as a parametrized curve in the plane. Let's write

$$\underbrace{f(t)}_{\text{input from } \mathbb{R}} = \underbrace{(x(t), y(t))}_{\text{output in the plane } \mathbb{R}^2}$$

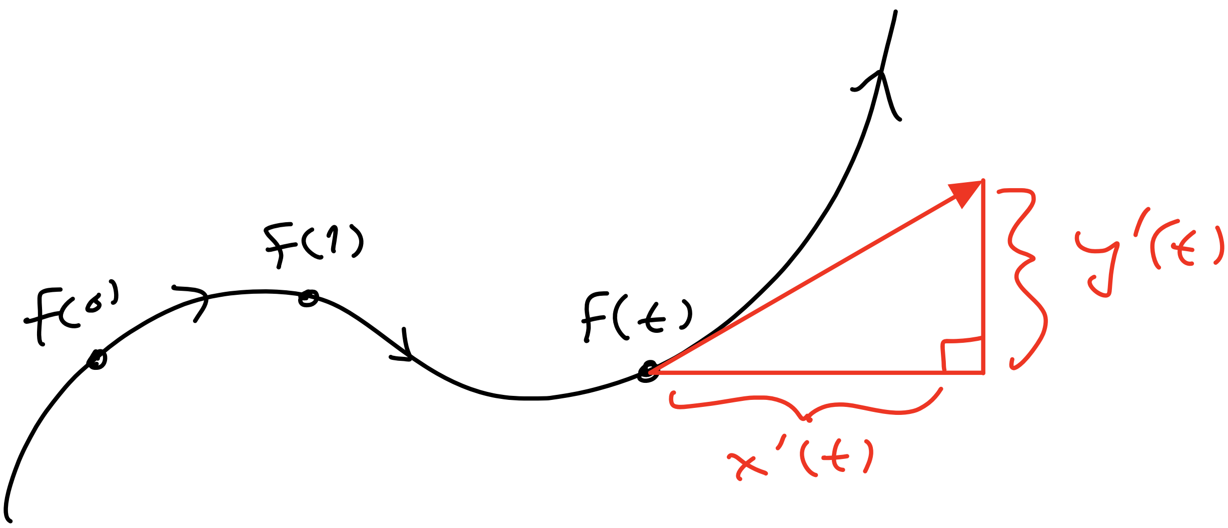
The derivative $f'(t)$ or df/dt is defined as

$$f'(t) = (x'(t), y'(t))$$

$$\frac{df}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

Note that $f' : \mathbb{R} \to \mathbb{R}^2$.

Picture:

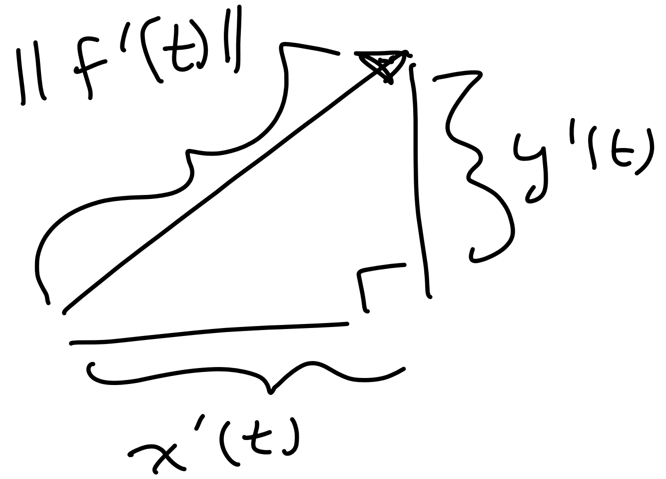


Every vector has a length or magnitude. The magnitude of velocity is speed.

$$\text{speed}(t) = \| f'(t) \|$$

$$= \sqrt{x'(t)^2 + y'(t)^2}.$$

Picture: Pythagorean Theorem.



$$\|f'(t)\|^2 = x'(t)^2 + y'(t)^2$$

Just as

$$\text{speed} = \frac{d}{dt}\,\text{distance},$$

we have

$$\text{distance} = \int \text{speed}\, dt$$

Arc length of path $f(t) = (x(t), y(t))$

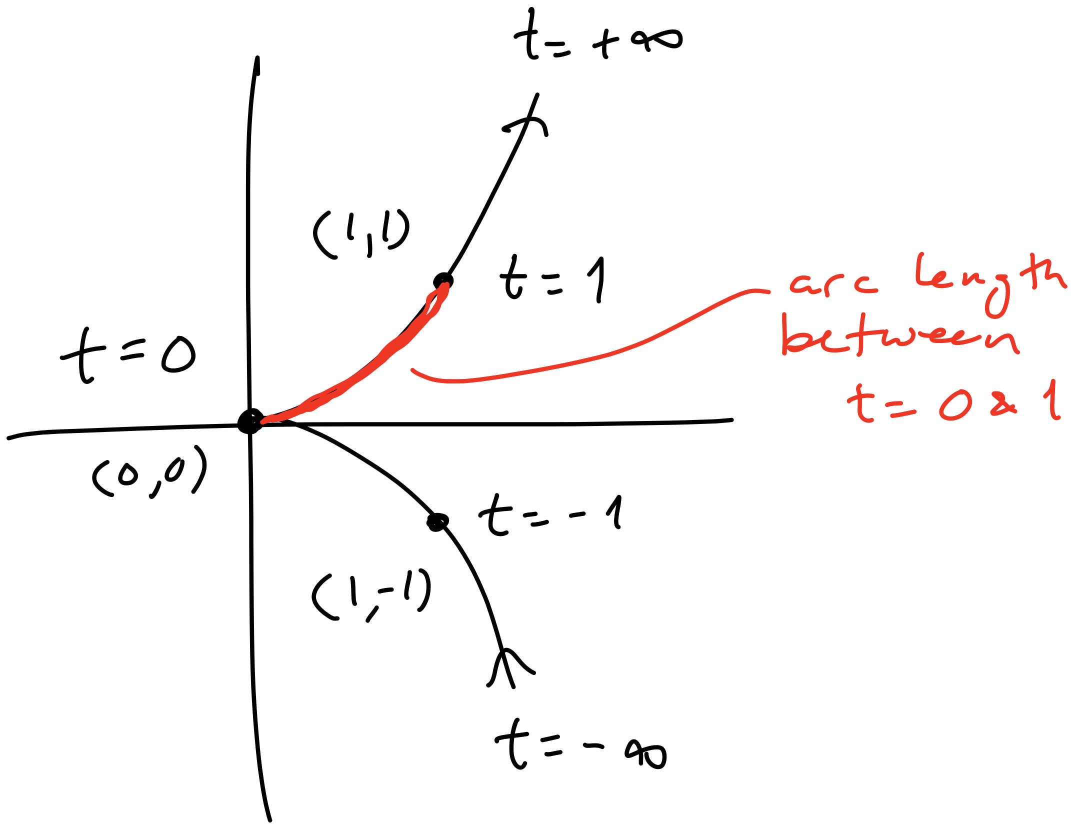
$$= \int \text{speed}\, dt = \int \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Generally, arc length integrals are <u>impossible</u> to solve by hand.

Here's a peculiar curve whose arc length <u>is</u> solvable by hand.

William Neile (1657):

$$f(t) = (t^2, t^3).$$



velocity $f'(t) = (2t, 3t^2)$

At time $t = 0$, velocity becomes

$(0,0)$. The particle briefly stops, then changes direction.

$$\text{speed} = \|\vec{r}\,'(t)\|$$

$$= \sqrt{(2t)^2 + (3t^2)^2}$$

$$= \sqrt{4t^2 + 9t^4}$$

$$= \sqrt{t^2(4+9t^2)}$$

$$= |t|\sqrt{4+9t^2}$$

Luckily this function can be integrated by hand.

Arc length

$$\int_{t=0}^{t=1} t\sqrt{4+9t^2}\,dt$$

$$\left[\,u = 4+9t^2, \quad du = 18\,t\,dt\,\right]$$

$$= \int_{u=4}^{u=13} \sqrt{u} \cdot \frac{du}{18} \qquad \int u^{1/2} = \frac{u^{3/2}}{3/2}$$

$$= \frac{1}{18} \cdot \frac{u^{3/2}}{3/2} \Bigg|_{u=4}^{u=13} \qquad \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{18}_{9}}$$

$$= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$$

$$\approx 1.439$$

It doesn't look nice but we did it by hand!

Chapter 2: Vectors

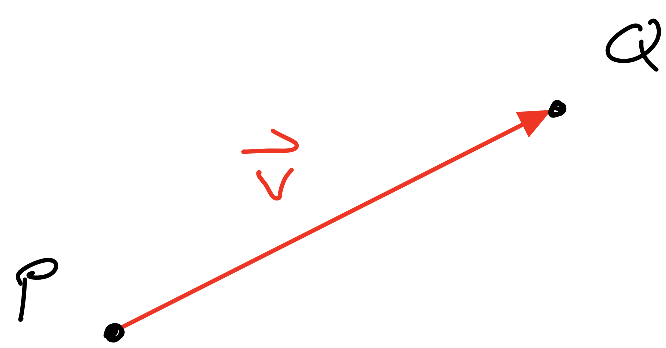
What is a "vector"?

- Physics: A vector is a

"quantity" with direction and magnitude.

- In this class, a vector is a directed line segment (an "arrow") in $\mathbb{R}^2$ or in $\mathbb{R}^3$.

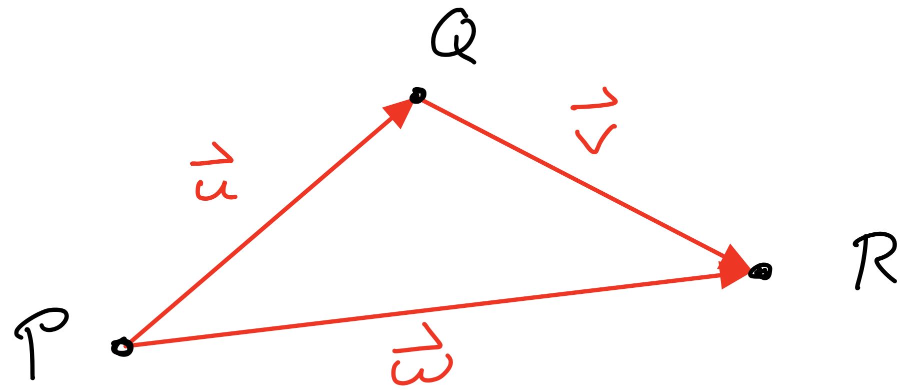
A vector is determined by an ordered pair of points P & Q, called the "tail" & "head":



Notation: $\vec{v} = \overrightarrow{PQ}$

There is an "arithmetic" of vectors. They can be <u>added</u> and <u>scaled by constants</u>.

Vectors are added "head-to-tail"



$$\vec{u} + \vec{v} = \vec{w}$$

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

But why do we call this "addition"?

Definition of "coordinates" (or "components" of a vector:

If $P = (x_1, y_1)$ & $Q = (x_2, y_2)$

then we write

$$\overrightarrow{PQ} = \langle x_2 - x_1 , y_2 - y_1 \rangle$$

emphasize that we are talking about a vector not a point.

Then addition of vectors becomes addition of components.

$$P = (x_1, y_1)$$
$$Q = (x_2, y_2)$$
$$R = (x_3, y_3)$$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$
$$\overrightarrow{QR} = \langle x_3 - x_2, y_3 - y_2 \rangle$$
$$\overrightarrow{PR} = \langle x_3 - x_1, y_3 - y_1 \rangle$$

Add component by component:

$$\overrightarrow{PQ} + \overrightarrow{QR} =$$

$$= \langle (\cancel{x_2} - x_1) + (x_3 - \cancel{x_2}),\ (\cancel{y_2} - y_1) + (y_3 - \cancel{y_2}) \rangle$$

$$= \langle x_3 - x_1,\ y_3 - y_1 \rangle$$

$$= \overrightarrow{PR} \quad \checkmark$$

Example: $P = (1, 1)$, $Q = (3, 5)$, $R = (6, 3)$.

$$\vec{u} = \overrightarrow{PQ} = \langle 3-1,\ 5-1 \rangle = \langle 2, 4 \rangle$$

$$\vec{v} = \overrightarrow{QR} = \langle 6-3,\ 3-5 \rangle = \langle 3, -2 \rangle$$

$$\vec{w} = \overrightarrow{PR} = \langle 6-1,\ 3-1 \rangle = \langle 5, 2 \rangle$$

Check:

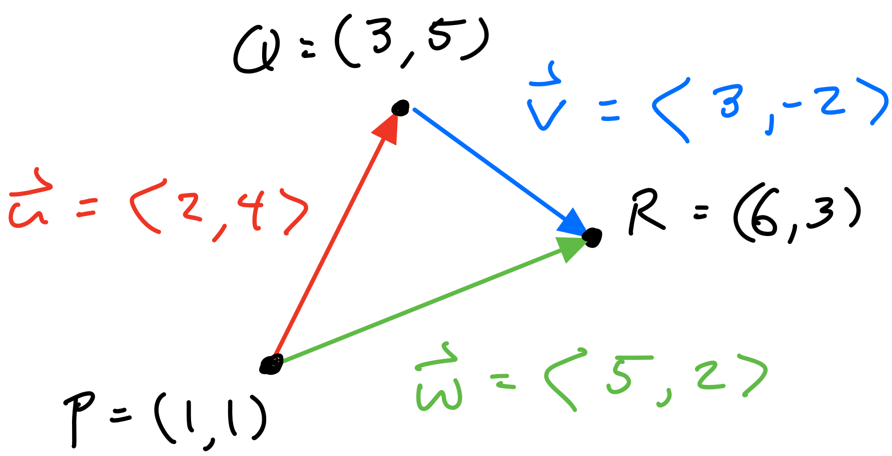
$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

$$\vec{u} + \vec{v} = \vec{w}$$

$$\langle 2, 4 \rangle + \langle 3, -2 \rangle$$

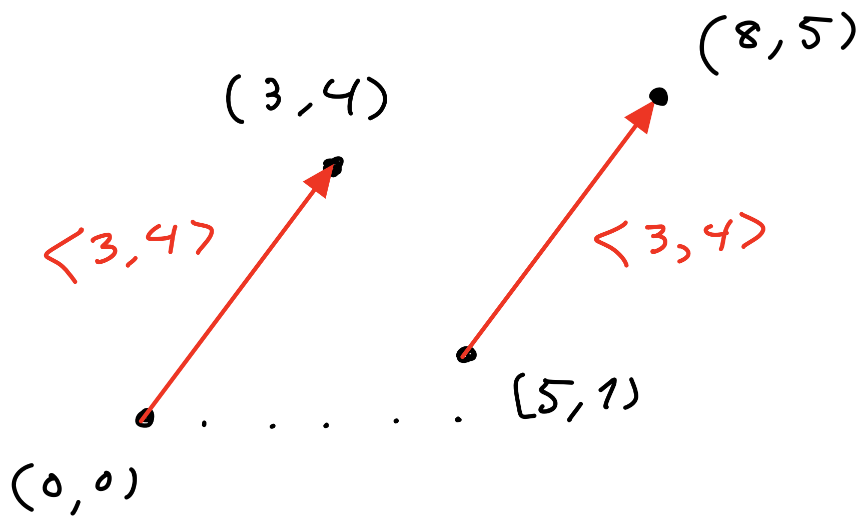
$$= \langle 2+3,\ 4+(-2) \rangle = \langle 5, 2 \rangle \quad \checkmark$$

Picture:



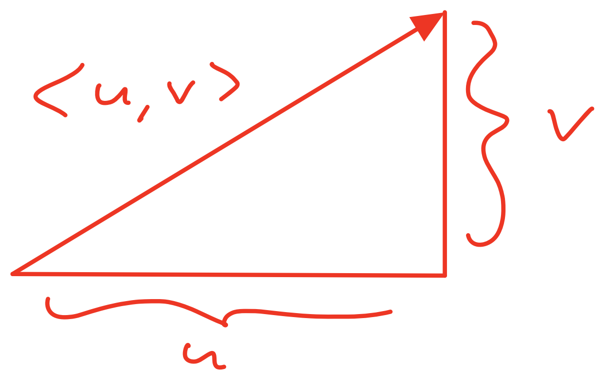
Subtlety: Just as a vector has magnitude & direction, two arrows should be considered "the same" when they have the same magnitude and direction.

e.g.



same vector in different locations.

To compute the magnitude we use the Pythagorean Theorem



$$\| \langle u, v \rangle \|^2 = u^2 + v^2$$

$$\| \langle u, v \rangle \| = \sqrt{u^2 + v^2}$$

e.g. $\| \langle 3, 4 \rangle \| = \sqrt{3^2 + 4^2}$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25} = 5.$$

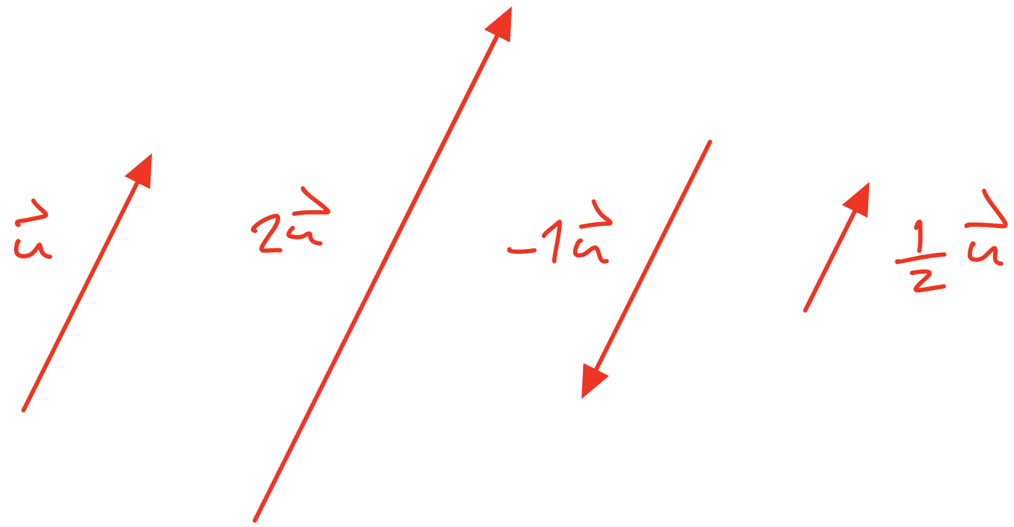
The other vector operation is called "scalar multiplication".

Given vector $\vec{u} = \langle u_1, u_2 \rangle$ and a number (scalar) k.

We define a new vector $k\vec{u}$ by multiplying each component by k:

$$k\vec{u} = \langle ku_1, ku_2 \rangle.$$

This changes the length but not the direction:



Finally, there is a special vector called the "zero vector", all of whose components are zero:

$$\vec{0} = \langle 0, 0 \rangle$$

[Tail & Head are equal.]

Here are the rules of vector arithmetic: (page 112 OpenStax)

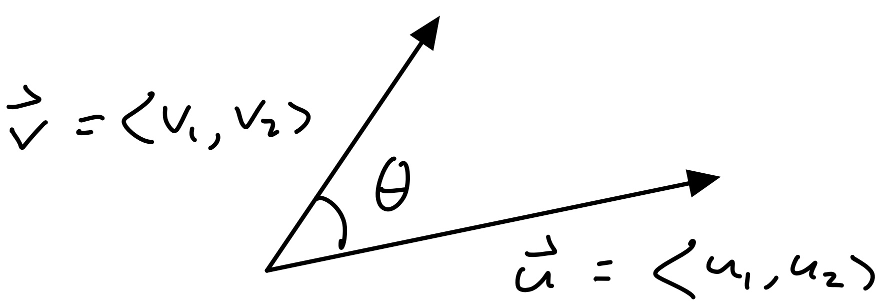
Consider vectors $\vec{u}, \vec{v}, \vec{w}$ and scalars r, s.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $r(s\vec{u}) = (rs)\vec{u}$
- $(r+s)\vec{u} = r\vec{u} + s\vec{u}$
- $r(\vec{u}+\vec{v}) = r\vec{u} + r\vec{v}$
- $1\vec{u} = \vec{u}$
- $0\vec{u} = \vec{0}$.

Moral: Everything that looks obvious is true!

Vector Arithmetic helps us to solve a hard problem: Find the angle between two vectors.

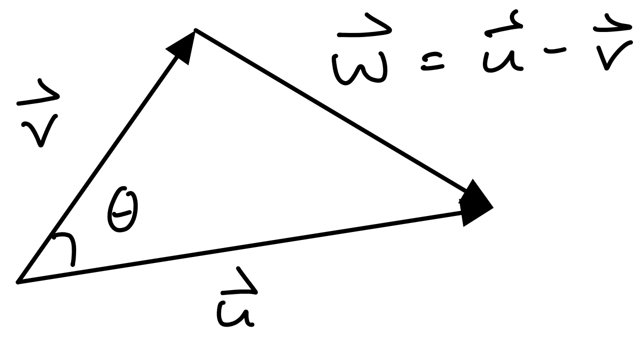


The angle is determined by the four numbers u_1, u_2, v_1, v_2 but what is the formula?

$$\theta = \text{some function of } u_1, u_2, v_1, v_2 \ldots$$

[The answer will involve a strange arithmetic operation called the "dot product" $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$]

Key: Draw a triangle:



$$\vec{v} + \vec{w} = \vec{u} \quad \text{so} \quad \vec{w} = \vec{u} - \vec{v}$$

The Law of Cosines says

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

On the other hand, we can compute the lengths in terms of the coordinates u_1, u_2, v_1, v_2.

[Details on HW 1]

The result says that

$$\boxed{u_1 v_1 + u_2 v_2 = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

Surprise !

We give this weird expression a name. We define

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 .$$

The Dot Product Theorem says

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta .$$

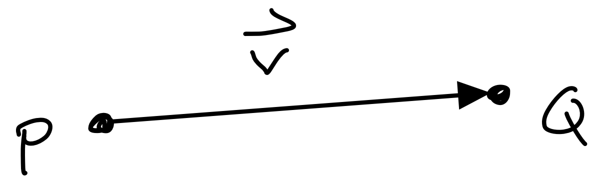
We can treat this as a third kind of arithmetic operation on vectors. It satisfies some rules :

(pg 147 of OpenStax)

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $s(\vec{u} \cdot \vec{v}) = (s\vec{u}) \cdot \vec{v} = \vec{u} \cdot (s\vec{v})$
- $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

HW1 is due Fri before class, on Blackboard.

A vector is an ordered pair of points in the plane (or in space of any # of dimensions). We view it as an arrow:



"$\vec{v} = \overrightarrow{PQ}$"

Coordinates: If $P = (p_1, p_2, p_3)$
$Q = (q_1, q_2, q_3)$

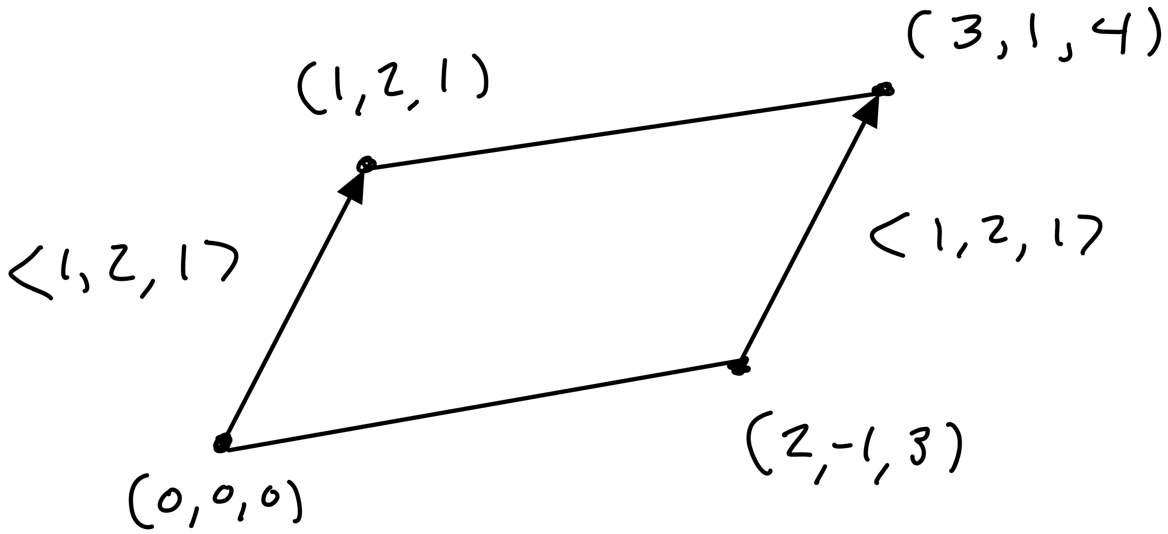
then write

$$\vec{v} = \overrightarrow{PQ} = \text{"head minus tail"}$$

$$= \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

Vectors can be moved around

without changing their coordinates:



Vectors with tail at (0,0,0) are in "standard position".

Sometimes we discuss vectors without mentioning the endpoints.

There are 3 basic operations:

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

define

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ (strange)

These operations satisfy quite a few "obvious rules".

[Remark: we have

vector + vector = vector

scalar · vector = vector

vector • vector = scalar

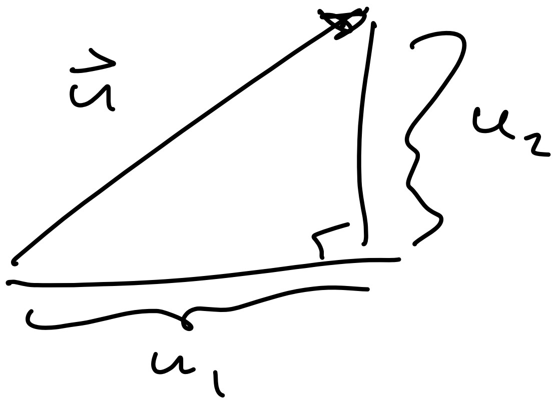
There is no general way to "multiply" vectors to get a vector:

vector × vector = vector ?

However, in 3D there is a strange operation called "cross product".]

Length (or magnitude) of a vector is computed using the Pythagorean theorem:

$$\vec{u} = \langle u_1, u_2 \rangle$$

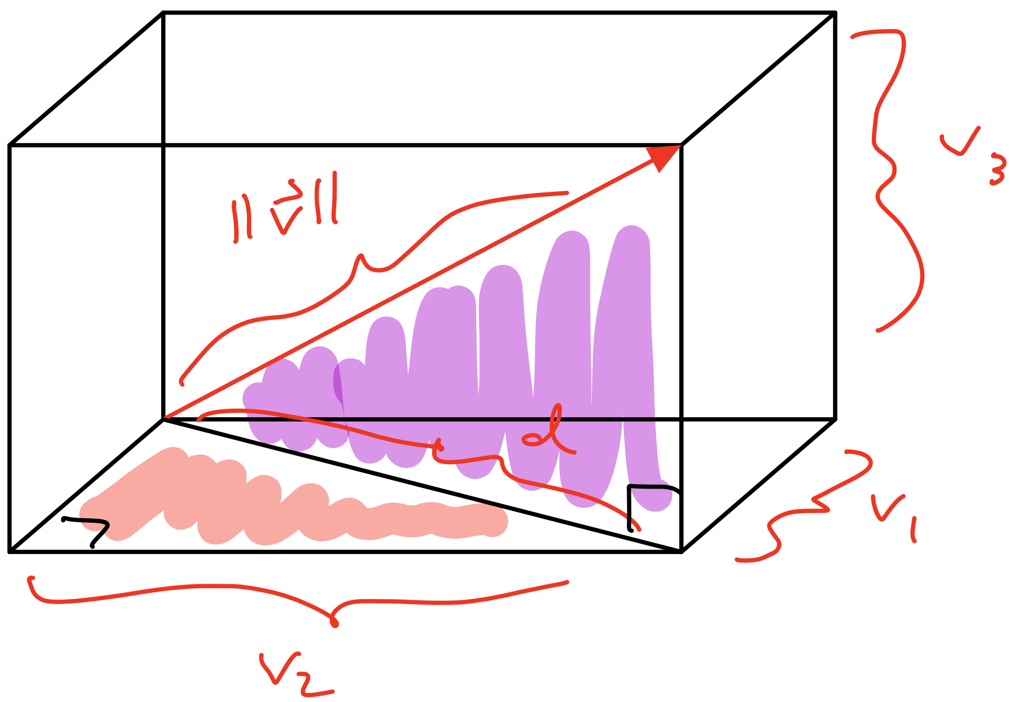


Let $\|\vec{u}\|$ be the length of $\vec{u}$.

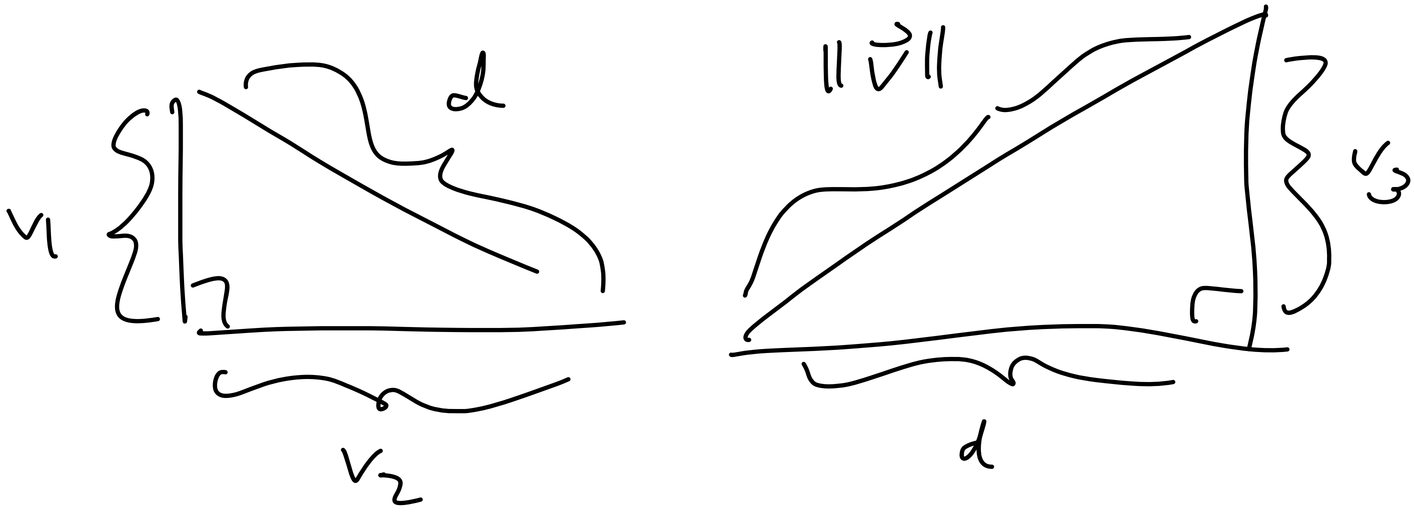
Then $\|\vec{u}\|^2 = u_1^2 + u_2^2$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$$

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$.



There are 2 right triangles here:



$$d^2 = v_1^2 + v_2^2 \qquad \|\vec{v}\|^2 = d^2 + v_3^2$$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

[By analogy: We use the same definition in any # of dimensions.

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

]

Dot Product is related to Lengths & Angles :

- Length :

$$\vec{v} \cdot \vec{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$$

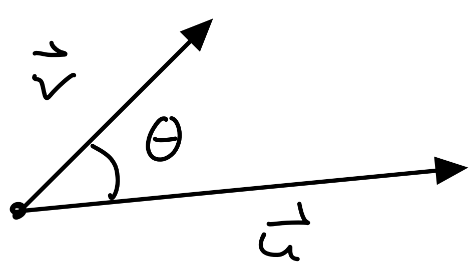
$$= v_1 v_1 + v_2 v_2 + v_3 v_3$$

$$= v_1^2 + v_2^2 + v_3^2$$

$$= \| \vec{v} \|^2$$

$$\boxed{\vec{v} \cdot \vec{v} = \| \vec{v} \|^2}$$

- Angles : Let θ be the angle between vectors $\vec{u}$ & $\vec{v}$ placed "tail-to-tail"



Dot Product Theorem:

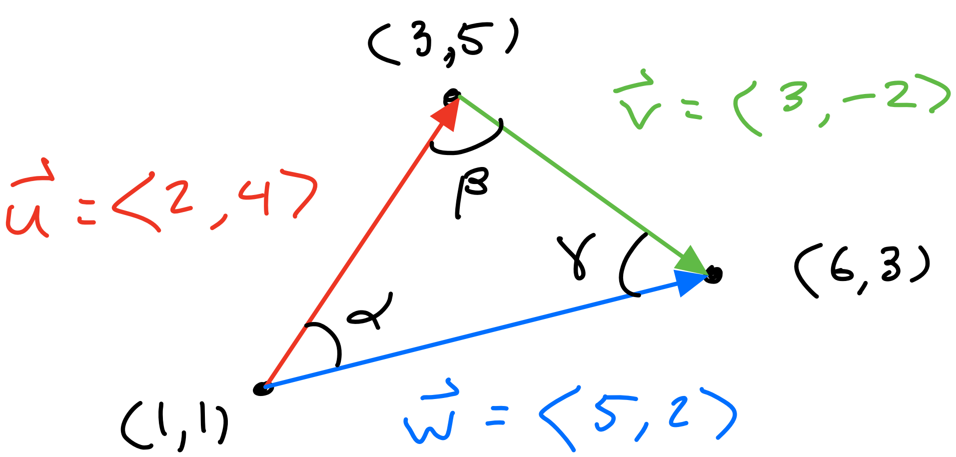
$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

Combine these boxed formulas:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\sqrt{\vec{u} \cdot \vec{u}} \cdot \sqrt{\vec{v} \cdot \vec{v}}}$$

Example: Consider the triangle.



Compute the angles α, β, γ.

First compute the dot products:

$$\vec{u} \cdot \vec{u} = 2^2 + 4^2 = 20 \longrightarrow \|\vec{u}\| = \sqrt{20}$$

$$\vec{v} \cdot \vec{v} = 3^2 + (-2)^2 = 13 \longrightarrow \|\vec{v}\| = \sqrt{13}$$

$$\vec{w} \cdot \vec{w} = 5^2 + 2^2 = 29 \longrightarrow \|\vec{w}\| = \sqrt{29}$$

$$\vec{u} \cdot \vec{v} = 2 \cdot 3 + 4 \cdot (-2) = -2$$

$$\vec{u} \cdot \vec{w} = 2 \cdot 5 + 4 \cdot 2 = 18$$

$$\vec{v} \cdot \vec{w} = 5 \cdot 3 + 2 \cdot (-2) = 11.$$

Compute α:

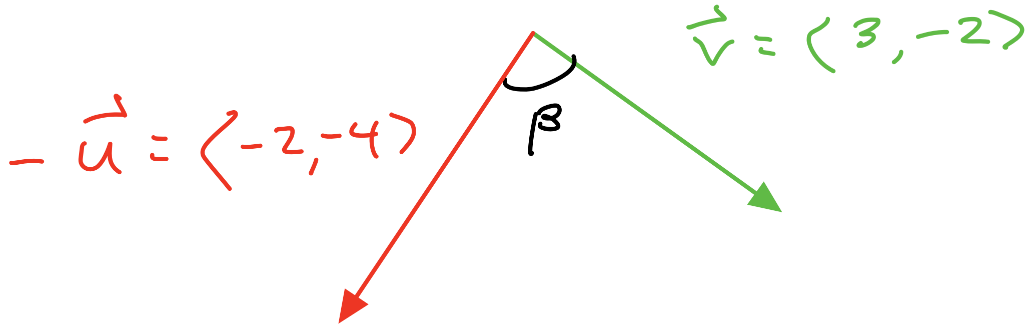
Since $\vec{u}$ & $\vec{w}$ are tail-to-tail:

$$\cos \alpha = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \, \|\vec{w}\|}$$

$$= \frac{18}{\sqrt{20} \cdot \sqrt{29}} \longrightarrow \alpha = 41.68^\circ$$

Compute β: Since $\vec{u}$ & $\vec{v}$ are not tail-to-tail,

Use $-\vec{u}$ instead of $\vec{u}$:



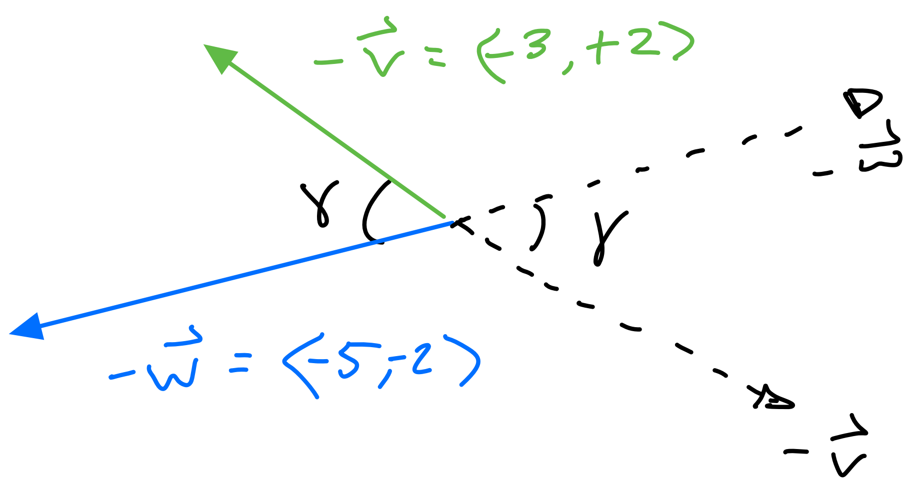
Observe : $\quad \| -\vec{u} \| = \| \vec{u} \| = \sqrt{20}$

$$(-\vec{u}) \cdot \vec{v} = -(\vec{u} \cdot \vec{v}) = +2$$

$$\cos \beta = \frac{(-\vec{u}) \cdot \vec{v}}{\| -\vec{u} \| \cdot \| \vec{v} \|} = \frac{2}{\sqrt{20} \cdot \sqrt{13}}$$

$$\longrightarrow \quad \beta = 82.85^\circ$$

Compute γ : Note $-\vec{v}$ & $-\vec{w}$ are tail to tail :



$$\cos \gamma = \frac{(-\vec{v}) \cdot (-\vec{w})}{\|-\vec{v}\| \cdot \|-\vec{w}\|}$$

$$= \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

$$\left[(-\vec{v}) \cdot (-\vec{w}) = -\left(\vec{v} \cdot (-\vec{w})\right) = \vec{v} \cdot \vec{w} \right]$$

$$= \frac{11}{\sqrt{13}\,\sqrt{29}}$$

$$\rightsquigarrow \quad \gamma = 55.49^\circ$$

Check:

$$\alpha + \beta + \gamma = 41.63^\circ + 88.85^\circ + 55.49^\circ$$

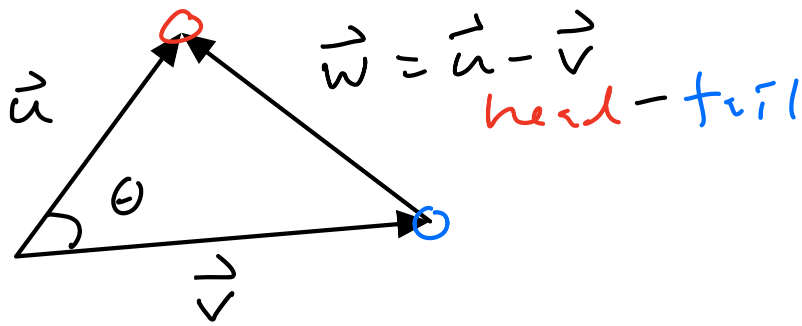
$$= 180^\circ \quad \checkmark$$

See HW 1 Problem 4 for an example in 3D.

HW 1 Problem 5 is about the

Dot Product Theorem:

Let θ be angle between vectors $\vec{u}$ & $\vec{v}$ placed tail to tail:



Let $\vec{w} = \vec{u} - \vec{v}$

$\vec{u} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$

Geometry:

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Algebra:

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$$

Don't think!

$$= \vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v} \cdot (\vec{u} - \vec{v})$$

[Compare $(a+b)c = ac + bc$]

$$= \cancel{\vec{u}\cdot\vec{u}} - \vec{u}\cdot\vec{v} - (\vec{v}\cdot\vec{u} - \cancel{\vec{v}\cdot\vec{v}})$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u}$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\,\vec{u}\cdot\vec{v}.$$

[use $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$]

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v}).$$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v})$$

Now you can think again. same.

Compare to the geometry:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

ALGEBRA $\longleftrightarrow$ GEOMETRY

Equations of Lines & Planes.

The gradeschool equation of a line is

$$y = mx + b.$$

But this equation does not generalize to higher dimensions.

In this class we prefer to write

$$ax + by = c.$$

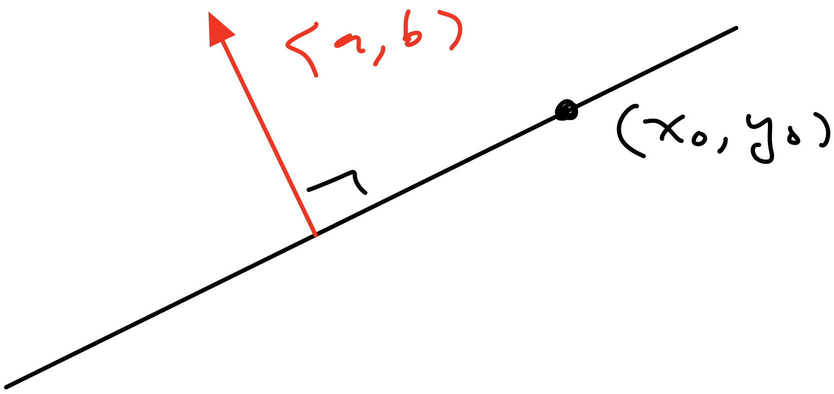
Even better:

$$\boxed{a(x - x_0) + b(y - y_0) = 0}$$

What does it mean?

I claim this is the line that

- passes through point (x_0, y_0)
- is perpendicular to vector $\langle a, b \rangle$.



Key: Dot product Theorem

$$\vec{u}\circ\vec{v} = \underbrace{\|\vec{u}\|\cdot\|\vec{v}\|}_{\neq 0}\cos\theta$$

Note that

$$\vec{u}\circ\vec{v} = 0 \iff \cos\theta = 0$$

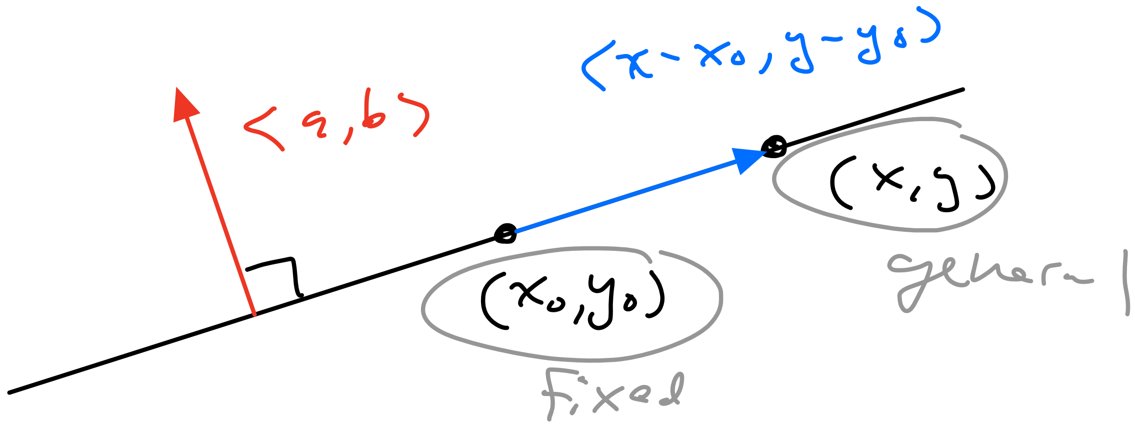
$$\iff \vec{u}\perp\vec{v}$$

"perpendicular"

$$\boxed{\vec{u}\circ\vec{v} = 0 \iff \vec{u}\perp\vec{v}}$$

So let L be the line passing through (x_0,y_0) and $\perp$ to the vector $\langle a,b\rangle$.

Then for any general point (x, y) on the line we have



Since the vector $\langle x-x_0, y-y_0 \rangle$ is parallel to L & $\langle a,b \rangle$ is $\perp$ to L we get

$$\langle a,b \rangle \perp \langle x-x_0, y-y_0 \rangle$$

hence

$$\langle a,b \rangle \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$a(x-x_0) + b(y-y_0) = 0 \quad \checkmark$$

We can convert into $y = mx + b$ form if we want.

Example: Consider the line in $\mathbb{R}^2$ passing through point $(1,2)$ & perp. to vector $\langle 3,1 \rangle$.

Let $(x_0,y_0) = (1,2)$

$\langle a,b \rangle = \langle 3,1 \rangle$

Equation of the line:

$$a(x-x_0) + b(y-y_0) = 0$$

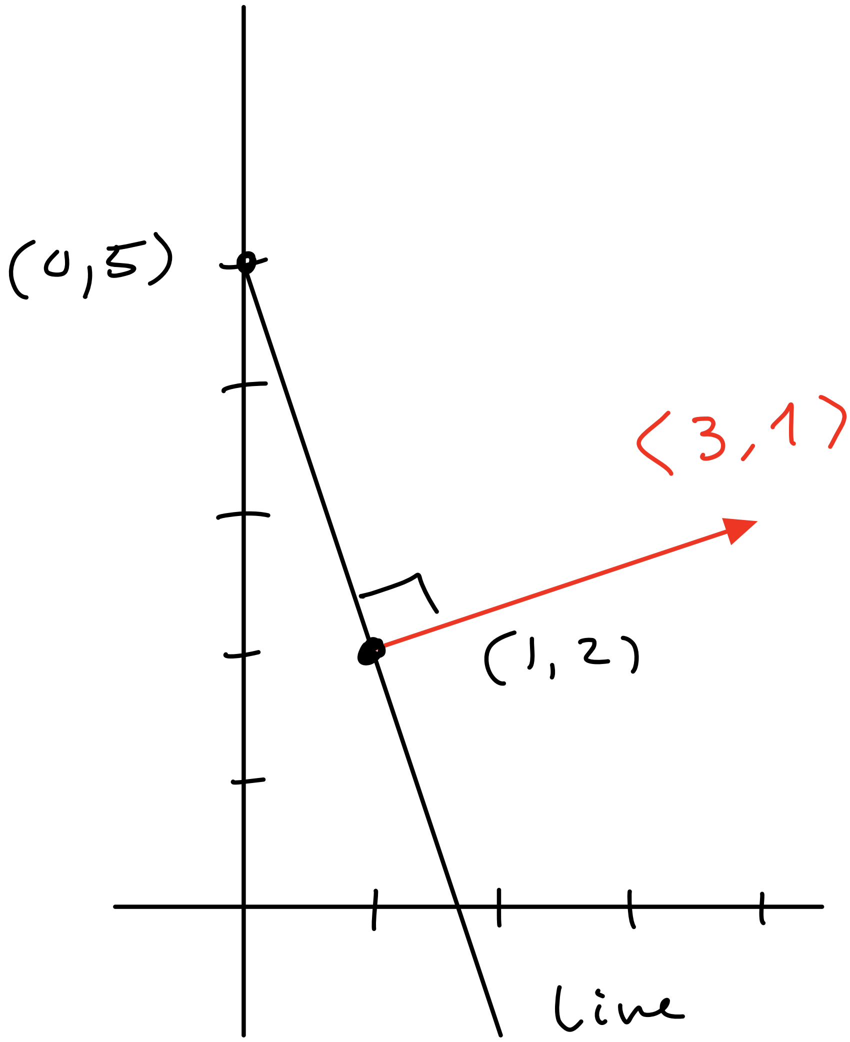
$$3(x-1) + 1(y-2) = 0$$

$$3x - 3 + y - 2 = 0$$

$$3x + y - 5 = 0$$

$$y = \underbrace{-3}_{\text{slope}}x + \underbrace{5}_{y\text{-intercept}}$$

Picture:



$$3(x-1)+1(y-2)=0$$

or

$$y=-3x+5$$

Note: The form of the equation is not unique. We could pick any point on the line

& any vector $\perp$ to line.

e.g. $(x_0, y_0) = (0, 5)$

$\langle a, b \rangle = \langle 6, 2 \rangle$

Check:

$$a(x - x_0) + b(y - y_0) = 0$$

$$6(x - 0) + 2(y - 5) = 0$$

$$6x - 0 + 2y - 10 = 0$$

$$6x + 2y - 10 = 0$$

divide by 2

$$3x + y - 5 = 0$$

$$y = -3x + 5$$

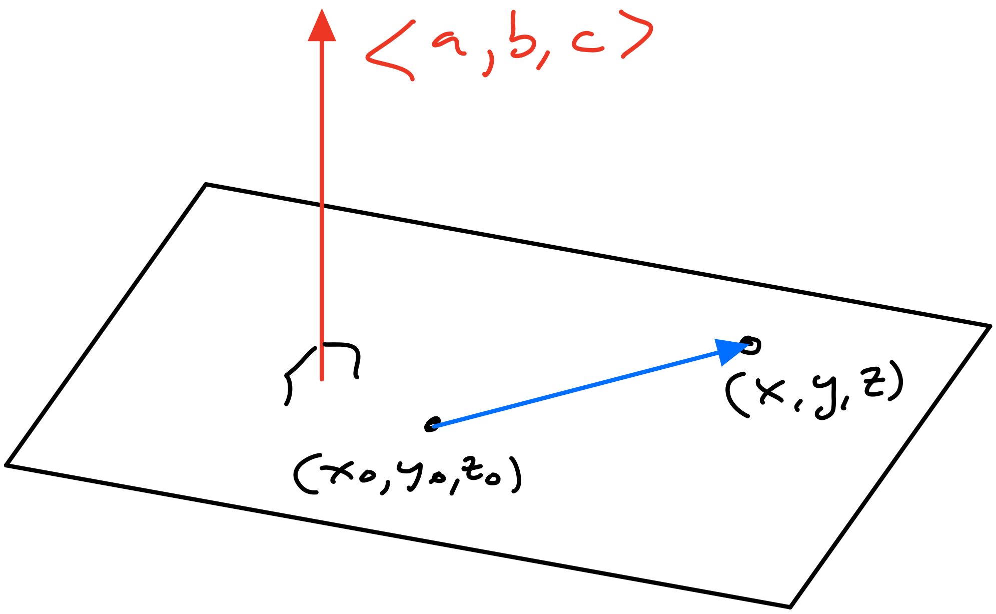
SAME ✓

Question: What shape is represented by following eq?

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is a <u>plane</u> in 3D !

Picture :



The plane contains (x_0, y_0, z_0) and is $\perp$ to vector $\langle a, b, c \rangle$ so for any point (x, y, z) in plane we have

$$\langle a, b, c \rangle \perp \langle x - x_0, y - y_0, z - z_0 \rangle$$

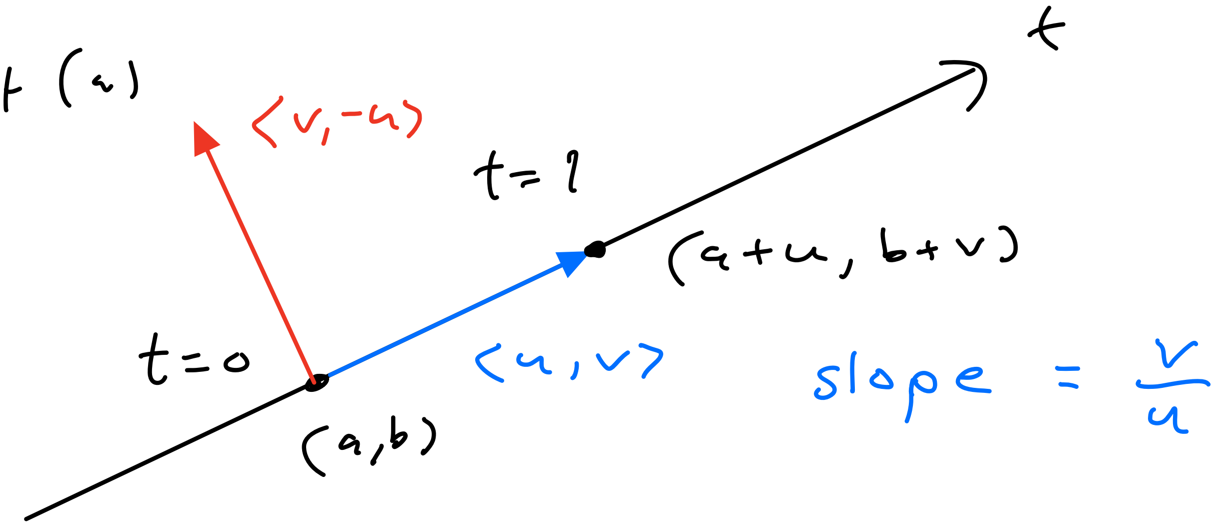
$$\underbrace{a(x - x_0) + b(y - y_0) + c(z - z_0)}_{\text{dot product.}} = 0$$

Today: HW 1 Discussion.

Monday: Quiz 1 at the beginning of class for 20 minutes (11:40AM – 12:00 PM)

Problem 1: Lines & Circles

Part (a)



This gives a parametrization

$$(x,y) = (a+ut,\ b+vt).$$

velocity

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (u,v)$$

CONSTANT VELOCITY!

Conversely: We will see later that any curve of constant velocity must be a straight line.

speed

$$\| \langle u, v \rangle \| = \sqrt{u^2 + v^2}$$

ALSO CONSTANT.

We can eliminate t:

$$x = a + ut \longrightarrow t = (x-a)/u$$
$$y = b + vt \longrightarrow t = (y-b)/v$$

$$\frac{(x-a)}{u} = \frac{(y-b)}{v} \quad \text{if } u \neq 0,\ v \neq 0$$

$$(x-a)v = (y-b)u \qquad \text{now } u = 0 \text{ or } v = 0 \text{ ok } \checkmark$$

$$vx - uy = av - bu$$

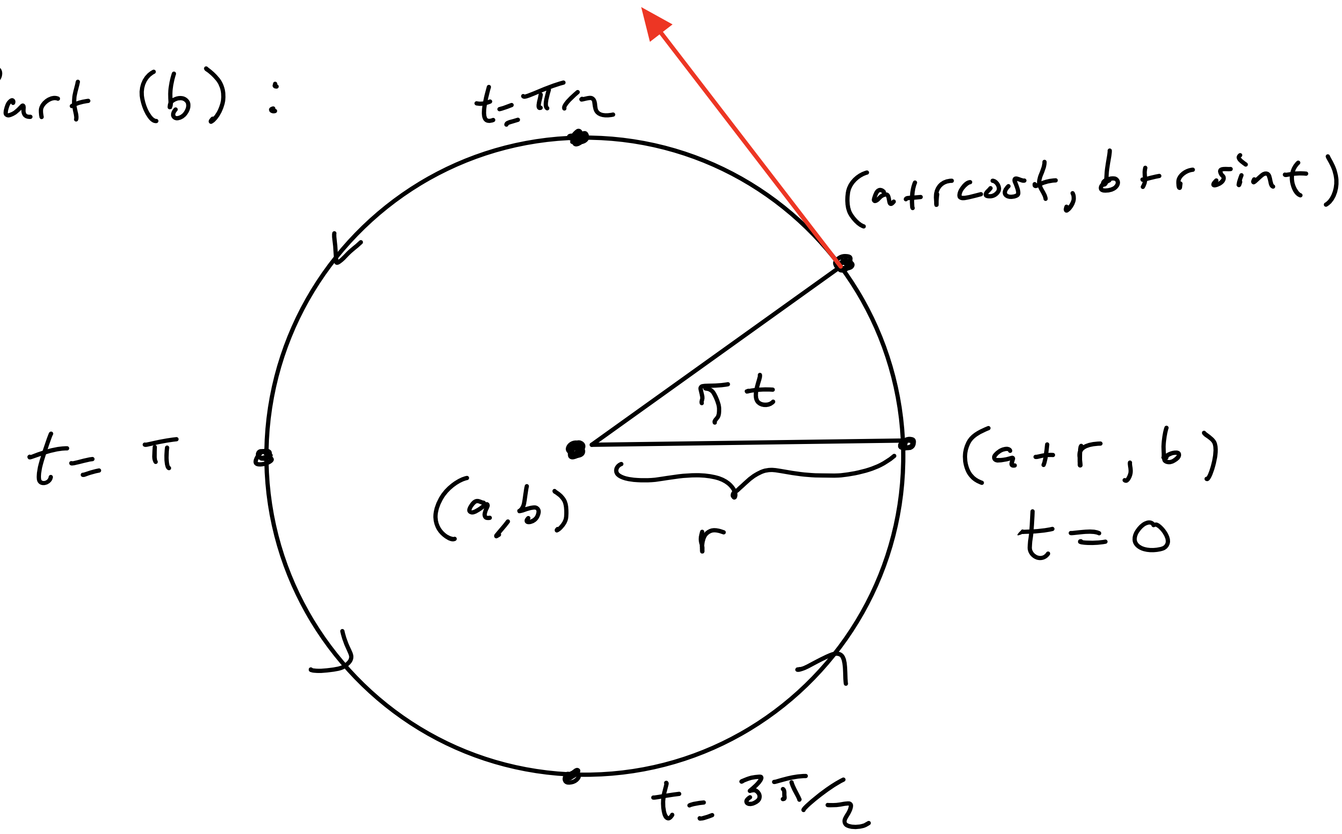
The equation of a line that is $\perp$ to vector $\langle v, -u \rangle$

Slope - Intercept:

$$uy = vx - av + bu$$

$$y = \underbrace{\frac{v}{u}}_{\text{slope}} x + b - \frac{av}{u}.$$

Part (b):



$$(x, y) = (a + r\cos t, \; b + r\sin t)$$

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-r\sin t, \; r\cos t)$$

NOT CONSTANT.

$$\text{speed} = \sqrt{(-r\sin t)^2 + (r\cos t)^2}$$

$$= \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= \sqrt{r^2 \left(\cancel{\sin^2 t + \cos^2 t}\right)}$$

$$= \sqrt{r^2} = |r| = r.$$

CONSTANT SPEED

We can eliminate t:

Use $\sin^2 t + \cos^2 t = 1$.

$$x = a + r\cos t \rightarrow \cos t = (x-a)/r$$

$$y = b + r\sin t \rightarrow \sin t = (y-b)/r$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x-a}{r}\right)^2 + \left(\frac{y-b}{r}\right)^2 = 1.$$

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle centred at (a,b) with radius r.

Problem 2:

$$(x,y) = (t^2-1,\ t^3-t)$$

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (2t,\ 3t^2-1)$$

slope of the tangent line at time t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{2t}$$

When is it $0, \pm 1, \infty$ (vertical)

$t = 0 \rightsquigarrow$ vertical tangent

$$(x,y) = (0^2-1,\ 0^3-0) = (-1, 0)$$

Slope 0: $\quad \dfrac{3t^2-1}{2t} = 0 \quad (t \neq 0)$

$$3t^2 - 1 = 0 \longrightarrow t = \pm\sqrt{1/3}$$

points:

$$(x,y) = \left(\frac{1}{3}-1,\ \left(\sqrt{\frac{1}{3}}\right)^3 - \sqrt{\frac{1}{3}}\right) \qquad t = \sqrt{\frac{1}{3}}$$

$$= (-0.67,\ 0.3)$$

$$(x,y) = \left(\frac{1}{3}-1, \left(-\sqrt{\frac{1}{3}}\right)^3 + \sqrt{\frac{1}{3}}\right)$$

$$= (-0.67, -0.3) \qquad t = -\sqrt{\frac{1}{3}}$$

Slope ± 1:

$$\frac{3t^2-1}{2t} = +1 \rightarrow 3t^2 - 1 = 2t$$

$$3t^2 - 2t - 1 = 0$$

$$t = \frac{2 \pm \sqrt{4+12}}{6}$$

$$= \frac{2 \pm 4}{6} = 1 \text{ or } -\frac{1}{3}$$

$(0,0)$ $(-.87, 0.3)$

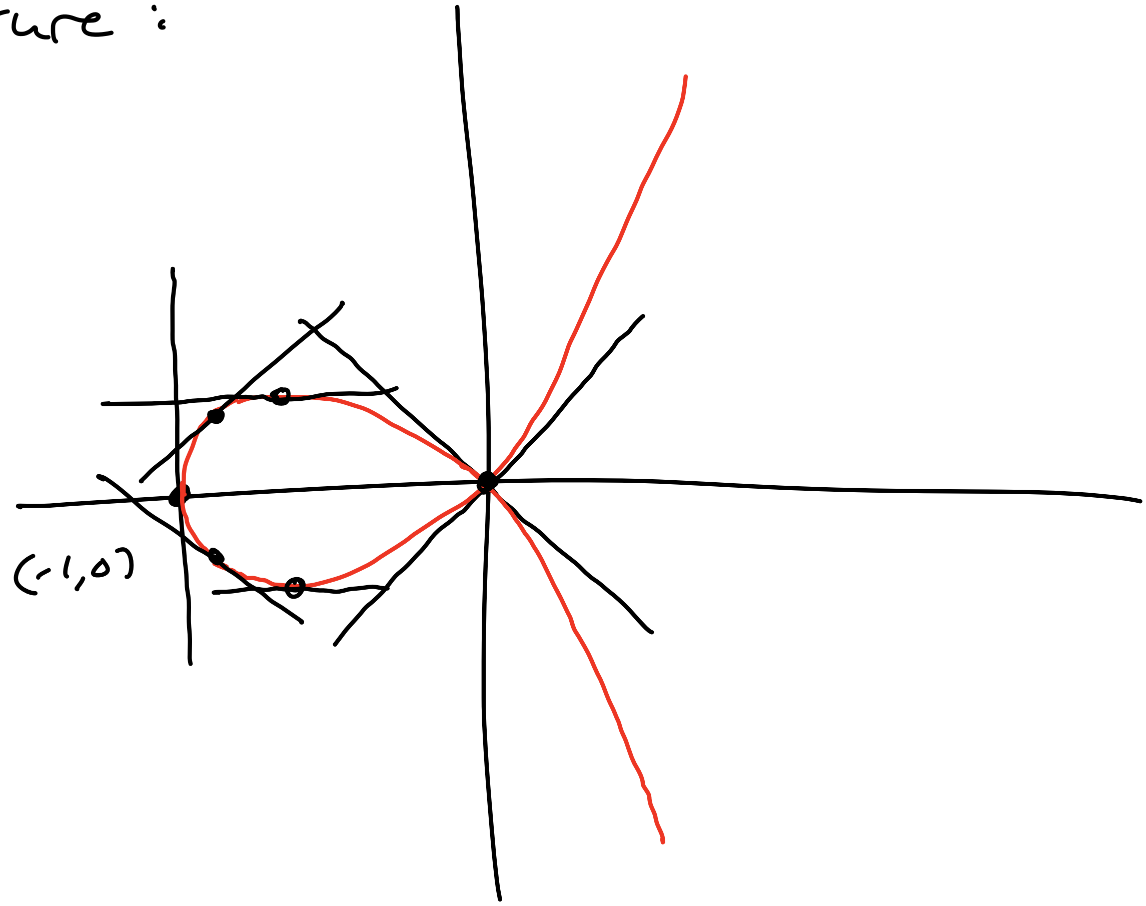
$$\frac{3t^2-1}{2t} = -1 \rightsquigarrow 3t^2 - 1 = -2t$$

$$3t^2 + 2t - 1 = 0$$

$$t = -1 \text{ or } +\frac{1}{3}$$

$(0,0)$ $(-0.87, -0.3)$

Picture:



Problem 3: The Cycloid.

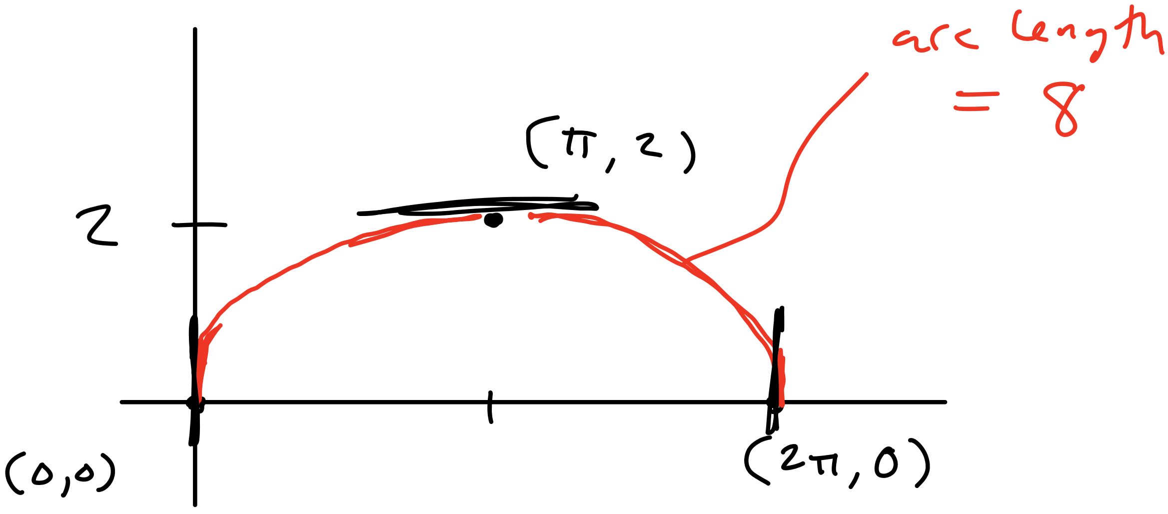
$$(x,y) = (t - \sin t,\ 1 - \cos t)$$

sketch between $t=0$, $t=2\pi$.

$t=0$: $(0 - \sin 0,\ 1 - \cos 0) = (0,0)$

$t=2\pi$: $(2\pi - \underbrace{\cancel{\sin 2\pi}}_{0},\ 1 - \underbrace{\cancel{\cos 2\pi}}_{1}) = (2\pi, 0)$

$t=\pi$: $(\pi - \underbrace{\cancel{\sin \pi}}_{0},\ 1 - \underbrace{\cancel{\cos \pi}}_{-1}) = (\pi, 2)$



slope of tangent $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$.

$t = \pi$: slope 0

$t = 0$ or $t = 2\pi$: slope $\to \infty$.

As $t \to 0$, $\frac{\sin t}{1-\cos t} \to \frac{0}{0}$ (oops!)

L'Hôpital's Rule

$$\lim_{t\to 0} \frac{\sin t}{1-\cos t} = \lim_{t\to 0} \frac{\cos t}{\sin t} = \infty$$

Arc Length:

velocity $= \langle 1-\cos t, \sin t \rangle$

$$\text{speed}^2 = (1-\cos t)^2 + \sin^2 t$$

$\Big[\ \sin^2 t = (\sin t)^2$. weird ... $\Big]$

$$= 1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1}$$

$$= 2 - 2\cos t$$

$$= 2(1-\cos t)$$

$$\text{speed} = \sqrt{2(1-\cos t)}$$

Something Lucky:

Half-Angle Formula

$$\sin\left(\frac{t}{2}\right) = \sqrt{\frac{1-\cos t}{2}}$$

$$\sin^2\left(\frac{t}{2}\right) = \frac{1-\cos t}{2}$$

$$1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$$

$$2(1-\cos t) = 4\sin^2\left(\frac{t}{2}\right).$$

So . . .

$$\text{speed} = \sqrt{2(1-\cos t)}$$

$$= \sqrt{4\sin^2(t/2)}$$

$$= 2\sin(t/2)$$

Finally: Arc length

$$= \int_0^{2\pi} \text{speed}\, dt$$

$$= \int_{t=0}^{t=2\pi} 2\sin\left(\frac{t}{2}\right) dt$$

$u = t/2$

$du = dt/2$

$dt = 2\,du.$

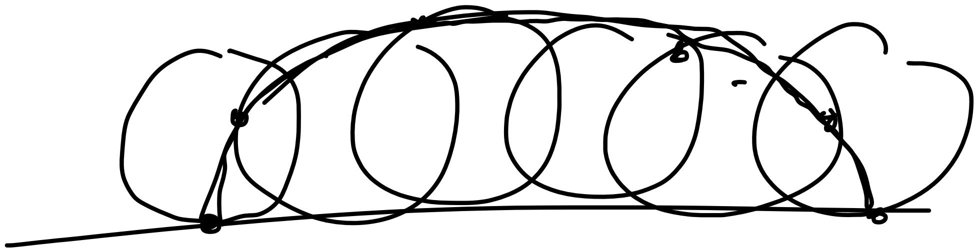
$$= \int_{u=0}^{u=\pi} 2\sin(u)\cdot 2\,du.$$

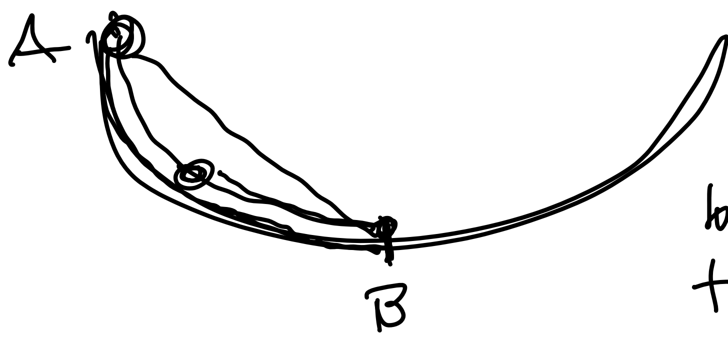
$$= 4\left[-\cos(u)\right]_0^{\pi}$$

$$= 4\left[\underbrace{\cancel{-\cos}(\pi)}_{1} + \underbrace{\cancel{\cos}(0)}_{1}\right]$$

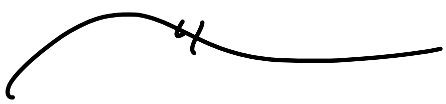
$$= 8 \quad \text{weird!}$$

Physics:

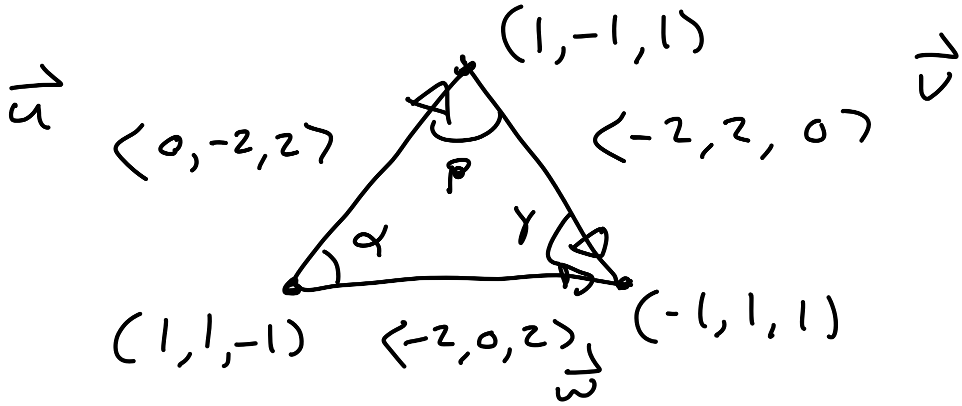




brachistochrone
tautochrone



Problem 4: Triangle in Space.



$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| =$

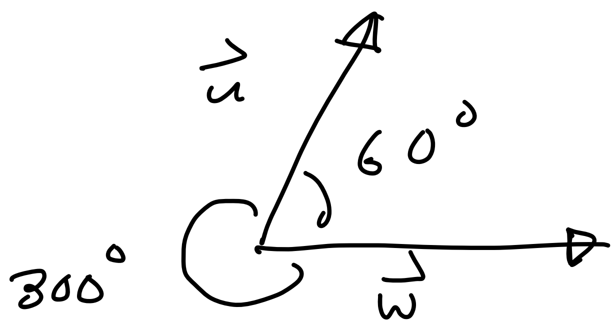
$$\sqrt{0^2+(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}.$$

$$\cos\alpha = \frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\,\|\vec{w}\|} = \frac{(0)(-2)+(-2)(0)+(2)(2)}{\sqrt{8}\cdot\sqrt{8}}$$

$$= \frac{4}{8} = \frac{1}{2}$$

$$\alpha = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \ \underline{\underline{\text{or}}}\ 300^\circ$$

pick the smaller.

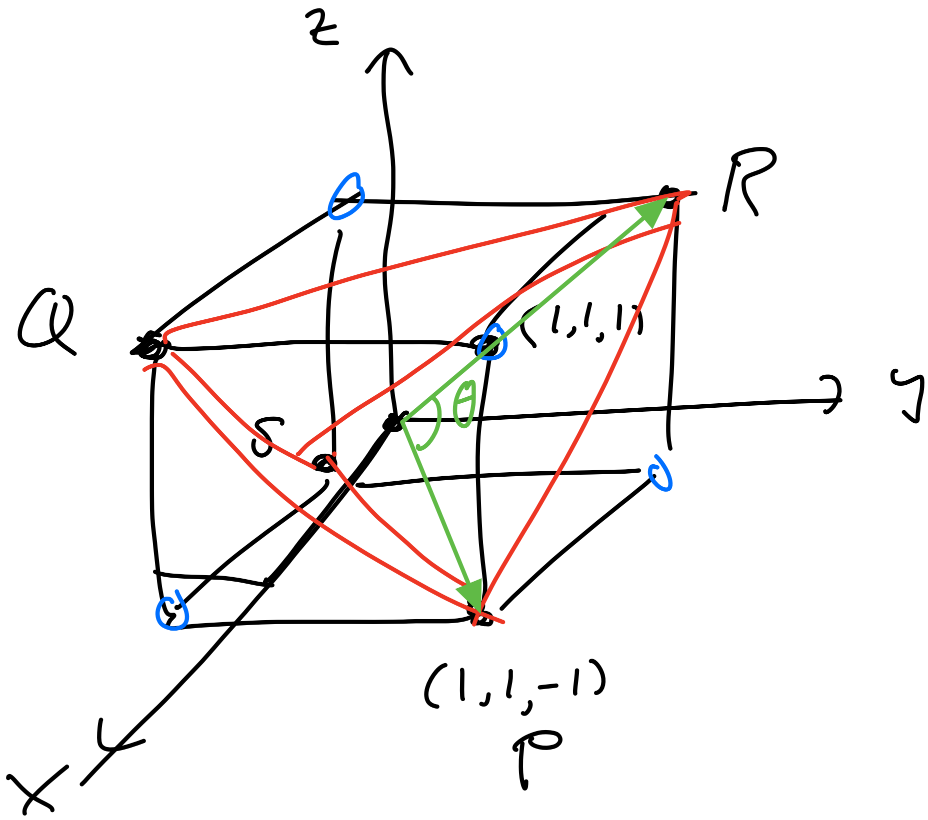


Remark: Add 4th point $S = (-1,-1,-1)$

Then PQR, PQS, PRS, QRS

all equilateral triangles

PQRS is a regular <u>tetrahedron</u>



Call the origin $O = (0,0,0)$

The tetrahedral angle θ between any two vertices, measured from the central point O.

θ = angle between $\overrightarrow{OP}$ & $\overrightarrow{OR}$

$$\cos\theta = \frac{\overrightarrow{OP} \cdot \overrightarrow{OR}}{\|\overrightarrow{OP}\| \, \|\overrightarrow{OR}\|}$$

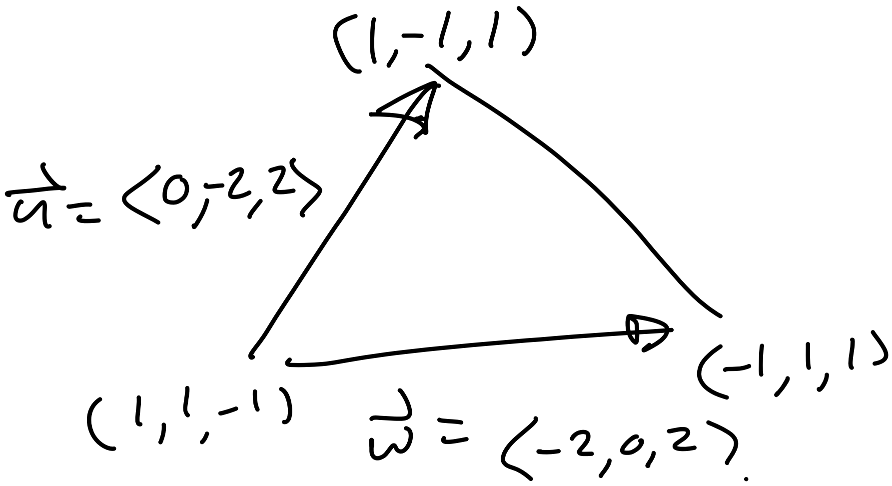
$$= \frac{\langle 1,1,-1\rangle \cdot \langle -1,1,1\rangle}{\|\langle 1,1,-1\rangle\| \, \|\langle -1,1,1\rangle\|}$$

$$= \frac{-1+1-1}{\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{3}$$

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^\circ$$

Problem 6: Too easy!

Here's a harder version. Find equation of the plane containing points P, Q, R from Problem 4.



Need one point ✓ & the normal vector.

To get a normal vector, take cross product of any two vectors in the plane. e.g. $\vec{u} = \langle -2,2,0\rangle$

$\vec{w} = \langle -2,0,2\rangle$.

$$\vec{u} \times \vec{w} = \text{“}\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 2 & 0\\ -2 & 0 & 2\end{pmatrix}\text{”}$$

$$= \vec{i} \det\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$$- \vec{j}\det\begin{pmatrix}-2&0\\-2&2\end{pmatrix}$$

$$+ \vec{k} \det\begin{pmatrix}-2&2\\-2&0\end{pmatrix}$$

$$= (4-0)\vec{i} - (-4-0)\vec{j} + 4\vec{k}$$

$$= 4\vec{i} + 4\vec{j} + 4\vec{k}$$

$$= \langle 4,4,4\rangle$$

Using normal vector $\langle 4,4,4 \rangle$ and any point, say $P = (1,1,-1)$, the equation of the plane is

$$4(x-1) + 4(y-1) + 4(z-(-1)) = 0$$

$$4x + 4y + 4z = 4$$

$$x + y + z = 1$$

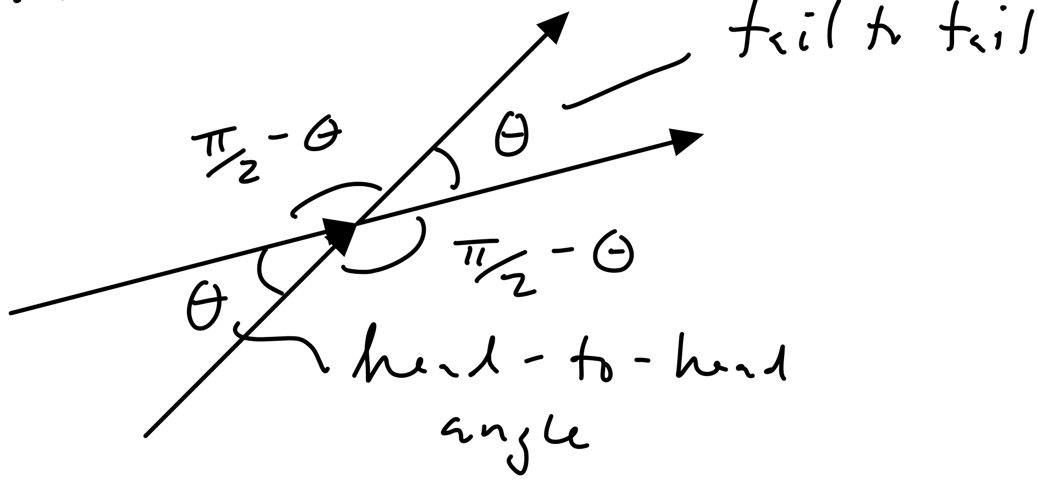
In retrospect this was very easy to see. Indeed the x, y, z coordinates of any point in the plane sum to 1.

$$P = (1,1,-1) \longrightarrow 1+1-1 = 1$$

$$Q = (1,-1,1) \longrightarrow 1-1+1 = 1$$

$$R = (-1,1,1) \longrightarrow -1+1+1 = 1$$

Remark :


tail to tail
θ
$\pi/2 - \theta$
$\pi/2 - \theta$
θ
head-to-head
angle