Math 310 **Spring 2024**
Homework 6 **Drew Armstrong**

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1. Various Kinds of First and Second Derivatives in $\mathbb{R}^3$. For any scalar field $f(x,y,z)$ we define a vector field Grad(f) and a scalar field Laplacian(f) by

$$\begin{aligned}\text{Grad}(f) &= \text{“}\nabla f\text{”} = \langle f_x, f_y, f_z\rangle,\\ \text{Laplacian}(f) &= \text{“}\nabla^2 f\text{”} = f_{xx} + f_{yy} + f_{zz}.\end{aligned}$$

and for any vector field $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ we define a vector field Curl($\mathbf{F}$) and a scalar field div($\mathbf{F}$) by

$$\begin{aligned}\text{Curl}(\mathbf{F}) &= \text{“}\nabla \times \mathbf{F}\text{”} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle,\\ \text{Div}(\mathbf{F}) &= \text{“}\nabla \bullet \mathbf{F}\text{”} = P_x + Q_y + R_z.\end{aligned}$$

(a) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that Curl(Grad(f)) $= \langle 0,0,0\rangle$.
(b) For any vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ check that Div(Curl($\mathbf{F}$)) $= 0$.
(c) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that Div(Grad(f)) $=$ Laplacian(f).

2. Conservative Vector Fields. Consider the vector field

$$\mathbf{F}(x,y,z) = \langle 2x + y, x + z, y\rangle.$$

(a) Check that the curl is zero: $\nabla \times \mathbf{F}(x,y,z) = \langle 0,0,0\rangle$.
(b) It follows from (a) that there exists a scalar field $f(x,y,z)$ satisfying $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$. Find one example of such a field. [Hint: Integrate $\mathbf{F}$ along an arbitrary path starting at some arbitrary point and ending at the point (x,y,z). For the purpose of this calculation let x,y,z be constant.]

3. Green's Theorem on a Rectangle. Consider the vector field $\mathbf{F}(x,y) = \langle y^2, x^2\rangle$.

(a) Compute the scalar curl of $\mathbf{F}$.
(b) Integrate the scalar curl of $\mathbf{F}$ over the rectangle with $0 \leq x \leq 2$ and $0 \leq y \leq 1$.
(c) Let C_1, C_2, C_3, C_4 be the four sides of the rectangle, oriented counterclockwise. Integrate $\mathbf{F}$ along each of these curves and add the results. Check that your answers to (a) and (b) are the same. [Hint: You can parametrize the four sides by

$$\begin{aligned}\mathbf{r}_1(t) &= (0,0) + t(2,0),\\ \mathbf{r}_2(t) &= (2,0) + t(0,1),\\ \mathbf{r}_3(t) &= (2,1) + t(-2,0),\\ \mathbf{r}_4(t) &= (0,1) + t(0,-1),\end{aligned}$$

each with $0 \leq t \leq 1$.]

4. Stokes' Theorem on a Pringle. Consider the constant vector field $\mathbf{F}(x,y,z) = \langle -y, x, 1\rangle$ and the pringle-shaped surface D defined by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^2\cos v\sin v\rangle,$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

(a) Compute the curl $\nabla \times \mathbf{F}(x,y,z)$.

(b) Compute the flux of the curl $\nabla \times \mathbf{F}$ across the pringle:

$$\iint_D (\nabla \times \mathbf{F}) \bullet \mathbf{N}\, dA = \iint (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv$$
$$= \iint (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\, dudv.$$

[Hint: From the previous homework we have $\mathbf{r}_u \times \mathbf{r}_v = \langle -u^2 \sin v, -u^2 \cos v, u\rangle$.]

(c) Let $C = \partial D$ be the boundary curve of the pringle. Compute the circulation of $\mathbf{F}$ around C. Check that your answers to (b) and (c) are the same. [Hint: If $\mathbf{r}(t)$ is a parametrization of C then the circulation is defined by

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \int \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt$$
$$= \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt.$$

You can take $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \sin t\rangle$ with $0 \le t \le 2\pi$. At the very end you will need the trig identity $2\cos^2 t = 1 + \cos(2t)$.]