Math 310 **Spring 2024**
Homework 5 **Drew Armstrong**

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1. Integrating a Scalar Along a Parabola. Let C be the portion of the parabola $y = x^2$ between points $(0,0)$ and $(1,1)$, and consider the scalar function $f(x,y) = x$. Compute the integral of f along C:

$$\int_C f\, ds.$$

[Hint: You must choose a parametrization of C. I recommend $\mathbf{r}(t) = (t, t^2)$ with $0 \leq t \leq 1$. The resulting integral may be computed by hand using substitution.]

Using the parametrization $\mathbf{r}(t) = (t, t^2)$ with $0 \leq t \leq 1$ we have

$$\begin{aligned}
\int_C f\, ds &= \int_0^1 f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt \\
&= \int_0^1 f(t, t^2)\|\langle 1, 2t\rangle\|\, dt \\
&= \int_0^1 t\sqrt{1 + 4t^2}\, dt \\
&= \frac{1}{8}\int_1^5 \sqrt{u}\, du && u = 1 + 4t^2, du = 8t\, dt \\
&= \frac{1}{8}\frac{u^{3/2}}{3/2}\Bigg|_{u=1}^{u=5} \\
&= \frac{1}{12}\left(5^{3/2} - 1\right).
\end{aligned}$$

2. Projection. Let $\mathbf{F}$ and $\mathbf{u}$ be any vectors with $\|\mathbf{u}\| = 1$ and let $\mathbf{p}$ be the *component of* $\mathbf{F}$ *in the direction of the unit vector* $\mathbf{u}$.

(a) Since $\mathbf{p}$ is parallel to $\mathbf{u}$ we know that $\mathbf{p} = t\mathbf{u}$ for some scalar t. Use the fact that the vector $\mathbf{p} - \mathbf{F}$ is perpendicular to $\mathbf{u}$ to prove that $t = \mathbf{F} \bullet \mathbf{u}$.

(b) Draw a picture showing the vectors $\mathbf{F}$, $\mathbf{u}$ and $\mathbf{p}$.

(a) We assume that $\mathbf{u}$ is a unit vector, so that $\mathbf{u} \bullet \mathbf{u} = 1$. Since $\mathbf{p} = t\mathbf{u}$ and since $\mathbf{p} - \mathbf{F}$ is perpendicular to $\mathbf{u}$ we have

$$\begin{aligned}
(\mathbf{p} - \mathbf{F}) \bullet \mathbf{u} &= 0 \\
\mathbf{p} \bullet \mathbf{u} - \mathbf{F} \bullet \mathbf{u} &= 0 \\
t\mathbf{u} \bullet \mathbf{u} &= \mathbf{F} \bullet \mathbf{u} \\
t &= \mathbf{F} \bullet \mathbf{u}.
\end{aligned}$$

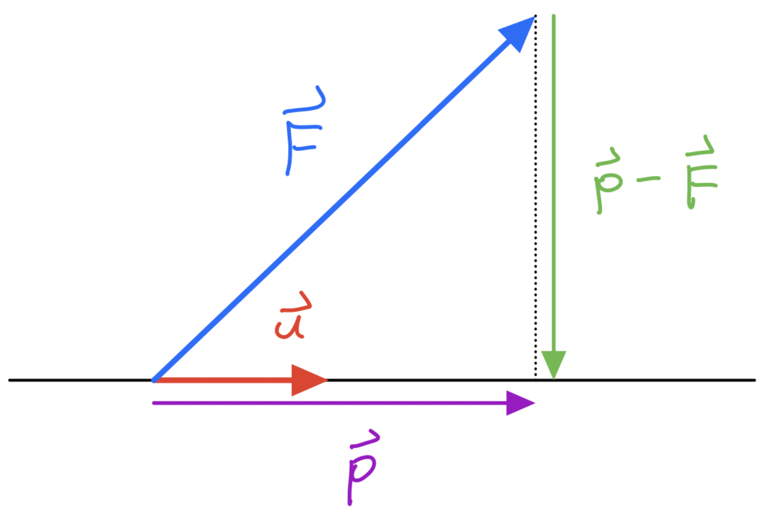
In other words, the projection of $\mathbf{F}$ onto $\mathbf{u}$ is $\mathbf{p} = t\mathbf{u} = (\mathbf{F} \bullet \mathbf{u})\mathbf{u}$.

Remark: For a general vector $\mathbf{v}$ we consider the unit vector $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$, which points in the same direction. Then the projection of $\mathbf{F}$ onto the direction of $\mathbf{v}$ is

$$(\mathbf{F} \bullet \mathbf{u})\mathbf{u} = \left(\mathbf{F} \bullet \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{\mathbf{F} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{F} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\right)\mathbf{v}.$$

Because this is more complicated, projections are usually computed with unit vectors.

(b) Picture:



3. Integrating Vector Fields Along Different Curves. By definition, the integral of a vector field $\mathbf{F}$ along a parametrized curve $\mathbf{r}(t)$ is

$$\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt.$$

Consider the two fields $\mathbf{F}(x, y) = \langle 2xy, x^2\rangle$, $\mathbf{G}(x, y) = \langle 2y, x^2\rangle$, and the two different curves $\mathbf{r}(t) = (t, t)$ and $\mathbf{s}(t) = (t, t^2)$ between the points $(0, 0)$ and $(1, 1)$.

(a) Integrate $\mathbf{F}$ along $\mathbf{r}(t)$ and $\mathbf{s}(t)$. Observe that you get the same answer.
(b) Integrate $\mathbf{G}$ along $\mathbf{r}(t)$ and $\mathbf{s}(t)$. Observe that you don't get the same answer.

(a) To integrate $\mathbf{F}(x, y) = \langle 2xy, x^2\rangle$ along $\mathbf{r}(t) = (t, t)$ we first compute $\mathbf{r}'(t) = \langle 1, 1\rangle$ and $\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(t, t) = \langle 2t^2, t^2\rangle$. Then we compute

$$\begin{aligned}
\int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt &= \int_0^1 \langle 2t^2, t^2\rangle \bullet \langle 1, 1\rangle\, dt \\
&= \int_0^1 (2t^2 + t^2)\, dt \\
&= \int_0^1 3t^2\, dt \\
&= 3\frac{1}{3}t^3\Big|_{t=0}^{t=1} \\
&= 1.
\end{aligned}$$

To integrate $\mathbf{F}$ along $\mathbf{s}(t) = (t, t^2)$ we first compute $\mathbf{s}'(t) = \langle 1, 2t\rangle$ and $\mathbf{F}(\mathbf{s}(t)) = \mathbf{F}(t, t^2) = \langle 2t^3, t^2\rangle$, hence

$$\begin{aligned}
\int_0^1 \mathbf{F}(\mathbf{s}(t)) \bullet \mathbf{s}'(t)\, dt &= \int_0^1 \langle 2t^3, t^2\rangle \bullet \langle 1, 2t\rangle\, dt \\
&= \int_0^1 (2t^3 + 2t^3)\, dt
\end{aligned}$$

$$= \int_0^1 4t^3\,dt$$
$$= 4\frac{1}{4}t^4\Big|_{t=0}^{t=1}$$
$$= 1.$$

Note that we get the same answer. This happened because $\mathbf{F}$ is a conservative vector field with potential $f(x,y) = x^2y$. Indeed:

$$\nabla f(x,y) = \langle f_x, f_y\rangle = 2xy, x^2\rangle = \mathbf{F}(x,y).$$

This implies that for **any** path from $(0,0)$ to $(1,1)$ we must have

$$\int_{(0,0)}^{(1,1)} \mathbf{F} = f(1,1) - f(0,0) = 1^2\cdot 1 - 0^2 \cdot 0 = 1.$$

(b) To integrate $\mathbf{G}(x,y) = \langle 2y, x^2\rangle$ along $\mathbf{r}(t) = (t,t)$ we first compute $\mathbf{r}'(t) = \langle 1,1\rangle$ and $\mathbf{G}(\mathbf{r}(t)) = \mathbf{G}(t,t) = \langle 2t, t^2\rangle$. Then we compute

$$\begin{aligned}\int_0^1 \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int_0^1 \langle 2t, t^2\rangle \bullet \langle 1,1\rangle\,dt \\ &= \int_0^1 (2t+t^2)\,dt \\ &= 2\frac{1}{2}t^2 + \frac{1}{3}t^3\Big|_{t=0}^{t=1} \\ &= 1 + \frac{1}{3} \\ &= 4/3\end{aligned}$$

To integrate $\mathbf{G}$ along $\mathbf{s}(t) = (t,t^2)$ we first compute $\mathbf{s}'(t) = \langle 1, 2t\rangle$ and $\mathbf{G}(\mathbf{s}(t)) = \mathbf{G}(t,t^2) = \langle 2t^2, t^2\rangle$, hence

$$\begin{aligned}\int_0^1 \mathbf{G}(\mathbf{s}(t)) \bullet \mathbf{s}'(t)\,dt &= \int_0^1 \langle 2t^2, t^2\rangle \bullet \langle 1,2t\rangle\,dt \\ &= \int_0^1 (2t^2+2t^3)\,dt \\ &= 2\frac{1}{3}t^3 + 2\frac{1}{4}t^4\Big|_{t=0}^{t=1} \\ &= \frac{2}{3} + \frac{1}{2} \\ &= 7/6.\end{aligned}$$

Note that we do **not** get the same answer. This implies that the vector field $\mathbf{G}$ is not conservative, meaning there is **no** scalar field $g(x,y)$ that satisfies $\nabla g = \mathbf{G}$.

Remark: We could also check this using Green's criterion. Let $G = \langle P, Q\rangle$ with $P = 2y$ and $Q = x^2$. Since $Q_x = 2x$ and $P_y = 2$ are not equal, the field $\mathbf{G}$ is not conservative. It may still happen accidentally that the integral of $\mathbf{G}$ along two specific paths with the same endpoints are equal (as happened in the original version of this problem), but it won't happen in general.

4. Area of a Pringle. Let D be the surface in $\mathbb{R}^3$ defined by $z = xy$ and $x^2 + y^2 \leq 1$, which looks like a pringle chip. We can parametrize this region by

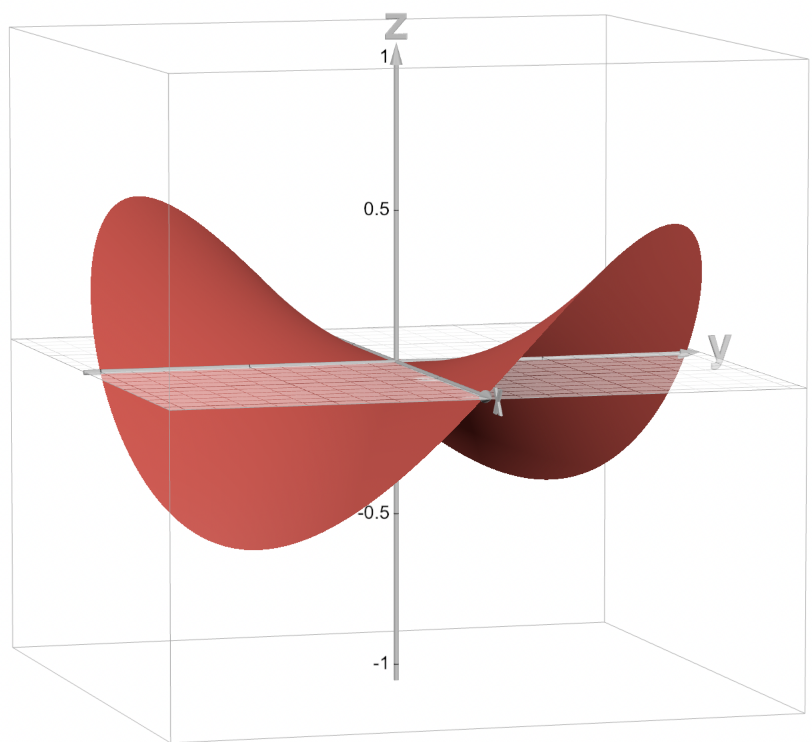
$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, (u\cos v)(u\sin v)\rangle$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. [Hint: Use the identity $\sin(2v) = 2\sin v\cos v$.]
(b) Compute the cross product $\mathbf{r}_u \times \mathbf{r}_v$.
(c) Compute the length $\|\mathbf{r}_u \times \mathbf{r}_v\|$ and simplify as much as possible. [Hint: The answer is $\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{u^2+1}$.]
(d) Use your answer from part (c) to compute the area of the pringle:

$$\text{Area}(D) = \iint_D 1\, \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv.$$

Here is a picture of the pringle:



(a) First we write

$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, u^2\sin(2v)/2\rangle.$$

Then we compute

$$\mathbf{r}_u = \langle \cos v, \sin v, u\sin(2v)\rangle,$$
$$\mathbf{r}_v = \langle -u\sin v, u\cos v, u^2\cos(2v)\rangle$$

(b) In order to simplify the cross product we will use the trig identities $\cos(2v) = \cos^2 v - \sin^2 v$ and $\sin(2v) = 2\sin v\cos v$. If $\mathbf{r}_u \times \mathbf{r}_v = \langle a, b, c\rangle$ the we have

$$a = (\sin v)(u^2\cos(2v)) - (u\sin(2v))(u\cos v)$$

$$
\begin{aligned}
&= (\sin v)(u^2(\cos^2 v - \sin^2 v)) - (2u \sin v \cos v)(u \cos v) \\
&= u^2 \sin v \cos^2 v - u^2 \sin^3 v - 2u^2 \sin v \cos^2 v \\
&= (u^2 \sin v)\left(\cos^2 v - \sin^2 v - 2\cos^2 v\right) \\
&= (u^2 \sin v)\left(-\cos^2 v - \sin^2 v\right) \\
&= (u^2 \sin v)(-1) \\
&= -u^2 \sin v,
\end{aligned}
$$

and

$$
\begin{aligned}
b &= (-u \sin v)(u \sin(2v)) - (\cos v)(u^2 \cos(2v)) \\
&= (-u \sin v)(2u \sin v \cos v) - (\cos v)(u^2(\cos^2 v - \sin^2 v) \\
&= -2u^2 \sin^2 v \cos v - u^2 \cos^3 v + u^2 \sin^2 v \cos v \\
&= (u^2 \cos v)(-2\sin^2 v - \cos^2 v + \sin^2 v) \\
&= (u^2 \cos v)(-\sin^2 v - \cos^2 v) \\
&= (u^2 \cos v)(-1) \\
&= -u^2 \cos v,
\end{aligned}
$$

and

$$
\begin{aligned}
c &= (\cos v)(u \cos v) - (\sin v)(-u \sin v) \\
&= u \cos^2 v + u \sin^2 v \\
&= u.
\end{aligned}
$$

In summary, we have

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -u^2 \sin v, -u^2 \cos v, u \rangle.$$

(c) The length of the cross product is

$$
\begin{aligned}
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{(-u^2 \sin v)^2 + (-u^2 \cos v)^2 + (u)^2} \\
&= \sqrt{u^4 \sin^2 v + u^4 \cos^2 v + u^2} \\
&= \sqrt{u^4 + u^2} \\
&= \sqrt{u^2(u^2+1)} \\
&= u\sqrt{u^2+1},
\end{aligned}
$$

because $u \geq 0$.

(d) Hence the surface area of the pringle is

$$
\begin{aligned}
\text{Area}(D) &= \int_D 1 \, dA \\
&= \iint 1 \|\mathbf{r}_u \times \mathbf{r}_v\| \, dudv \\
&= \iint u\sqrt{u^2+1} \, dudv \\
&= \int_0^{2\pi} dv \cdot \int_0^1 u\sqrt{u^2+1} \, du
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi \int_0^1 u\sqrt{u^2+1}\,du \\
&= \pi \int_1^2 \sqrt{w}\,dw && w = u^2+1, dw = 2u\,du \\
&= \pi \frac{w^{3/2}}{3/2}\bigg|_{w=1}^{w=2} \\
&= \frac{2\pi}{3}\left(2^{3/2}-1\right) \\
&\approx 3.83.
\end{aligned}
$$

Note that this is slightly larger than the area of a flat circle of radius 1 (i.e., 3.14).

5. Proof of Conservation of Energy. A *conservative force field* $\mathbf{F}$ has the form $\mathbf{F} = -\nabla f$ for some *scalar potential* f. Suppose that a particle of mass m travels along a trajectory $\mathbf{r}(t)$. Newton says that the force $\mathbf{F}$ acting on the particle satisties

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t).$$

The *kinetic energy* of the particle at time t is $\mathrm{KE}(t) = \frac{1}{2}m\|\mathbf{r}'(t)\|^2$, the *potential energy* at time t is $\mathrm{PE}(t) = f(\mathbf{r}(t))$, and the *total mechanical energy* is $E(t) = \mathrm{KE}(t) + \mathrm{PE}(t)$. Use the chain rule and product rule for derivatives to show that

$$E'(t) = 0.$$

[Hint: Write $\|\mathbf{r}'(t)\|^2 = \mathbf{r}'(t) \bullet \mathbf{r}'(t)$.]

First we use the "multivariable product rule" compute the derivative of $\mathrm{KE}(t)$:

$$
\begin{aligned}
\mathrm{KE}'(t) &= \left[\frac{1}{2}m\mathbf{r}'(t) \bullet \mathbf{r}'(t)\right]' \\
&= \frac{1}{2}m\left[\mathbf{r}'(t) \bullet \mathbf{r}'(t)\right]' \\
&= \frac{1}{2}m\left[\mathbf{r}''(t) \bullet \mathbf{r}'(t) + \mathbf{r}'(t) \bullet \mathbf{r}''(t)\right] \\
&= \frac{1}{2}m\left[2\mathbf{r}''(t) \bullet \mathbf{r}'(t)\right] \\
&= m\mathbf{r}''(t) \bullet \mathbf{r}'(t) \\
&= \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t).
\end{aligned}
$$

Then we use the "multivariable chain rule" to compute the derivative of $\mathrm{PE}(t)$:

$$
\begin{aligned}
\mathrm{PE}'(t) &= [f(\mathbf{r}(t))]' \\
&= \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \\
&= -\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t).
\end{aligned}
$$

Then putting the two together gives

$$E'(t) = \mathrm{KE}'(t) + \mathrm{PE}'(t) = \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) - \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = 0.$$

Thus the total mechanical energy of the system is constant.

6. Application of Conservation of Energy. Choose a coordinate system in $\mathbb{R}^3$ with the sun at position $(0,0,0)$. Suppose that the sun has mass M. If $\mathbf{F}(x,y,z)$ is the gravitational force exerted by the sun on a spaceship of mass m at position (x,y,z), Newton tells us that $\mathbf{F}(x,y,z) = -\nabla f(x,y,z)$, where[1]

$$f(x,y,z) = -1 \cdot \frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

is called the *gravitational potential*. At a certain time, the spaceship has speed s_0 and distance d_0 from the origin. At a later time the spaceship has speed s_1 and distance d_1 from the origin. Use conservation of energy to compute s_1 in terms of s_0, d_0 and d_1. (Assume that no other forces are acting on the spaceship.)

If the spaceship follows trajectory $\mathbf{r}(t)$ we note that

$$\mathrm{PE}(t) = f(\mathbf{r}(t)) = -\frac{GMm}{\|\mathbf{r}(t)\|}.$$

That is, the gravitational potential energy only depends on the distance of the spaceship from the origin. On the other hand, the kinetic energy only depends on the length of the velocity:

$$\mathrm{KE}(t) = \frac{1}{2}m\|\mathbf{r}'(t)\|^2.$$

Suppose the spacehsip has initial speed $s_0 = \|\mathbf{r}'(0)\|$ and initial distance $d_0 = \|\mathbf{r}(0)\|$, so the total mechanical energy at time zero is

$$E(0) = \frac{1}{2}ms_0^2 - \frac{GMm}{d_0}.$$

If s_1 and d_1 are the speed and distance at some other time (say $t = 1$), then the total mechanical energy at that time is

$$E(1) = \frac{1}{2}ms_1^2 - \frac{GMm}{d_1}.$$

Then since $E(t)$ is constant we must have

$$\begin{aligned} E(0) &= E(1) \\ \frac{1}{2}s_0^2 - \frac{GMm}{d_0} &= \frac{1}{2}s_1^2 - \frac{GMm}{d_1} \\ s_0^2 - \frac{2GM}{d_0} &= s_1^2 - \frac{2GM}{d_1}, \end{aligned}$$

which implies that

$$s_1 = \sqrt{s_0^2 + 2GM\left(\frac{1}{d_1} - \frac{1}{d_0}\right)}.$$

[1] G is a constant of nature called the *gravitational constant*.

Fun Application (Escape Velocity). The same formula works with a planet instead of the sun. Suppose that the spaceship sits on the surface of a planet of radius R and mass M, so that $d_0 = R$. What initial speed is needed to just escape gravitational field of the planet?

At some later time we want $s_1 = 0$ and $d_1 = \infty$, so by conservation of energy we must have

$$s_0 = \sqrt{s_1^2 + 2GM\left(\frac{1}{d_0} - \frac{1}{d_1}\right)} = \sqrt{0^2 + 2GM\left(\frac{1}{R} - \frac{1}{\infty}\right)} = \sqrt{\frac{2GM}{R}}.$$

The mass of the spacecraft is irrelevant, and the direction of launch is irrelevant (but don't launch it directly into the ground). Of course we neglect air resistance and any other forces.

To compute the escape velocity of the Earth, we input

$$\begin{aligned} G &= 6.673 \times 10^{-11} && \mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}, \\ R &= 6.271 \times 10^{6} && \mathrm{m}, \\ M &= 5.972 \times 10^{24} && \mathrm{kg}, \end{aligned}$$

to get

$$s_0 = \sqrt{\frac{2(6.673)10^{-11}(5.972)10^{24}}{(6.271)10^6}} \approx 1.13 \quad \mathrm{km/s}.$$