Math 310 **Spring 2024**
Homework 4 **Drew Armstrong**

Problem 1. Area of a Parametrized Region. Given a region D in $\mathbb{R}^2$, the area is

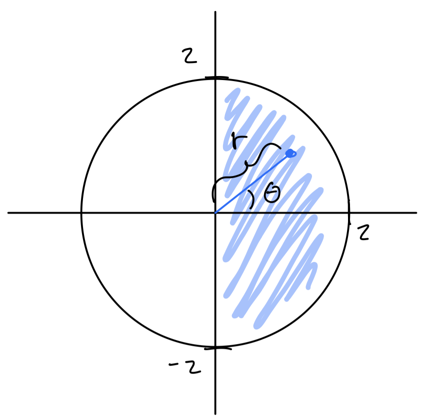
$$\text{Area}(D) = \iint_D 1\,dxdy.$$

For each of the following problems you should (1) draw the region, (2) find a parametrization, (3) use your parametrization to compute the area.

(a) The half-circle satisfying $x^2 + y^2 \leq 4$ and $x \geq 0$. [Hint: Use polar coordinates.]
(b) The region satisfying $x^2 + y^2 \leq 4$ and $x \geq 1$. [Hint: Don't use polar coordinates. You will need the antiderivative

$$\int 2\sqrt{4-x^2}\,dx = x\sqrt{4-x^2} + 4\arcsin(x/2).]$$

(a): In this case D is the right half of a circle of radius 2:

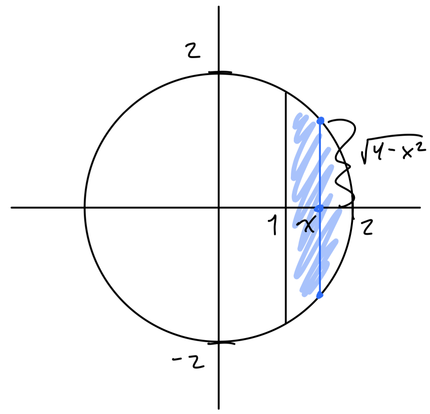


We parametrize D using polar coordinates with $0 \leq r \leq 2$ and $-\pi/2 \leq \theta \leq \pi/2$. The area is

$$\begin{aligned}
\iint 1\,dxdy &= \iint r\,drd\theta \\
&\int_{-\pi/2}^{\pi/2} 1\,d\theta \int_0^2 r\,dr \\
&= [\theta]_{\theta=-\pi/2}^{\theta=\pi/2} \left[\frac{1}{2}r^2\right]_{r=0}^{r=2} \\
&= \left[\frac{\pi}{2} + \frac{\pi}{2}\right]\left[\frac{1}{2}2^2\right] \\
&= 2\pi.
\end{aligned}$$

Note that this agrees with the formula $\pi(2)^2 = 4\pi$ for the area of the full circle.

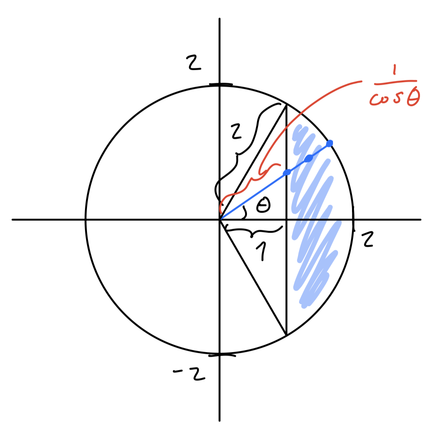
(b): Here is a picture of the region D:



We can parametrize this region by $1 \le x \le 2$ and $-\sqrt{4-x^2} \le x \le \sqrt{4-x^2}$. The area is

$$\begin{aligned}
\iint 1\, dxdy &= \int_1^2 \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1\, dy \right) dx \\
&= \int_1^2 [y]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\, dx \\
&= \int_1^2 2\sqrt{4-x^2}\, dx \\
&= \left[x\sqrt{4-x^2} + 4\arcsin(x/2) \right]_{x=1}^{x=2} \\
&= 2\sqrt{4-2^2} + 4\arcsin(2/2) - 1\sqrt{4-1^2} - 4\arcsin(1/2) \\
&= 0 + 4\arcsin(1) - \sqrt{3} - 4\arcsin(1/2) \\
&= 0 + 4(\pi/2) - \sqrt{3} - 4(\pi/6) \\
&= 4\pi/3 - \sqrt{3}.
\end{aligned}$$

Remark: It is also possible (but more difficult) to parametrize this region using polar coordinates. Consider the following picture:



The picture shows that $-\pi/3 \le \theta \le \pi/3$ and $1/\cos\theta \le r \le 2$, so the area is

$$\iint 1\, dxdy = \iint r\, drd\theta$$

$$
\begin{aligned}
&= \int_{-\pi/3}^{\pi/3} \left(\int_{1/\cos\theta}^{2} r\,dr \right) d\theta \\
&= \int_{-\pi/3}^{\pi/3} \left[\frac{1}{2} r^2 \right]_{r=1/\cos\theta}^{r=2} d\theta \\
&= \int_{-\pi/2}^{\pi/3} \left[2 - \frac{1}{2\cos^2\theta} \right] d\theta \\
&= \left[2\theta - \frac{\tan\theta}{2} \right]_{\theta=-\pi/3}^{\theta=\pi/3} \\
&= 2\left[\frac{\pi}{3} + \frac{\pi}{3} \right] - \frac{1}{2}\left[\tan(\pi/3) - \tan(-\pi/3) \right] \\
&= 2\left[\frac{2\pi}{3} \right] - \frac{1}{2}\left[\sqrt{3} + \sqrt{3} \right] \\
&= 4\pi/3 - \sqrt{3}.
\end{aligned}
$$

The fact that we got the same answer each time means that the calculations are probably correct. This problem can also be solved without Calculus:

`https://en.wikipedia.org/wiki/Circular_segment#Arc_length_and_area`

Problem 2. Center of Mass of a 2D Region. Let D be the region parametrized by $0 \leq x \leq 2$ and $x \leq y \leq 5x - 2x^2$. Think of D as a solid with mass density 1.

(a) Compute the total mass $M = \iint_D 1\,dxdy$.
(b) Compute the moments $M_x = \iint_D x\,dxdy$ and $M_y = \iint_D y\,dxdy$.
(c) Compute the center of mass.
(d) Draw the region and its center of mass.

(a): The mass (i.e., the area) of the region D is

$$
\begin{aligned}
M &= \iint_D 1\,dxdy \\
&= \int_0^2 \left(\int_x^{5x-2x^2} 1\,dy \right) dx \\
&= \int_0^2 \left(5x - 2x^2 - x \right) dx \\
&= \int_0^2 (4x - 2x^2)\,dx \\
&= \left[4\frac{1}{2}x^2 - 2\frac{1}{3}x^3 \right]_0^2 \\
&= 4\frac{1}{2}2^2 - 2\frac{1}{3}2^3 \\
&= 8/3.
\end{aligned}
$$

(b): The moment in the x direction is

$$
M_x = \iint_D x\,dxdy
$$

$$
\begin{aligned}
&= \int_0^2 \left(\int_x^{5x-2x^2} x \, dy \right) dx \\
&= \int_0^2 [xy]_{y=x}^{y=5x-2x^2} \, dx \\
&= \int_0^2 \left[x(5x - 2x^2) - x^2 \right] dx \\
&= \int_0^2 \left[4x^2 - 2x^3 \right] dx \\
&= \left[4\frac{1}{3}x^3 - 2\frac{1}{4}x^4 \right]_0^2 \\
&= 4\frac{1}{3}2^3 - 2\frac{1}{4}2^4 \\
&= 8/3.
\end{aligned}
$$

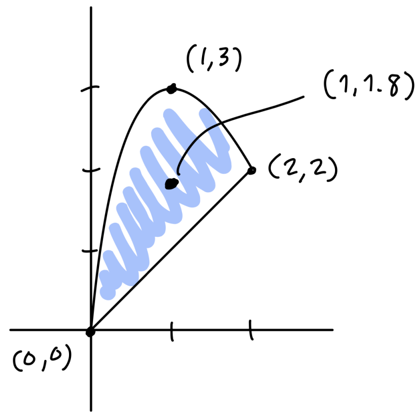
It is just a coincidence that $M_x = M$. The moment in the y direction is

$$
\begin{aligned}
M_y &= \iint_D y \, dxdy \\
&= \int_0^2 \left(\int_x^{5x-2x^2} y \, dy \right) dx \\
&= \int_0^2 \left[\frac{1}{2}y^2 \right]_{y=x}^{y=5x-2x^2} dx \\
&= \frac{1}{2} \int_0^2 \left[(5x - 2x^2)^2 - x^2 \right] dx \\
&= \frac{1}{2} \int_0^2 \left[4x^4 - 20x^3 + 24x^2 \right] dx \\
&= \frac{1}{2} \left[4\frac{1}{5}x^5 - 20\frac{1}{4}x^4 + 24\frac{1}{3}x^3 \right]_0^2 \\
&= \frac{1}{2} \left(4\frac{1}{5}2^5 - 20\frac{1}{4}2^4 + 24\frac{1}{3}2^3 \right) \\
&= 24/5.
\end{aligned}
$$

(c): The center of mass is

$$
(\bar{x}, \bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M} \right) = \left(\frac{8/3}{8/3}, \frac{24/5}{8/3} \right) = \left(1, \frac{9}{5} \right) = (1, 1.8).
$$

(d): Here is a picture:



Problem 3. Polar Coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. We already know that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = r.$$

The general theory predicts that we must also have

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \frac{1}{r}.$$

Check that this is true. [Hint: $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$.]

First we compute the derivatives using the one variable chain rule:

$$\begin{aligned} r_x &= (1/2)(x^2+y^2)^{-1/2}(2x) = x(x^2+y^2)^{-1/2}, \\ r_x &= (1/2)(x^2+y^2)^{-1/2}(2y) = y(x^2+y^2)^{-1/2}, \\ \theta_x &= 1/(1+(y/x)^2)(-y/x^2), \\ \theta_y &= 1/(1+(y/x)^2)(1/x). \end{aligned}$$

The formulas for θ_x and θ_y can be simplified using $1/(1+(y/x)^2) = x^2/(x^2+y^2)$ to get

$$\begin{aligned} \theta_x &= x^2/(x^2+y^2)(-y/x^2) = -y/(x^2+y^2), \\ \theta_y &= x^2/(x^2+y^2)(1/x) = x/(x^2+y^2). \end{aligned}$$

Then we compute the determinant:

$$\begin{aligned} \det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} &= r_x\theta_y - -r_y\theta_x \\ &= x(x^2+y^2)^{-1/2}x/(x^2+y^2) + y(x^2+y^2)^{-1/2}y/(x^2+y^2) \\ &= x^2(x^2+y^2)^{-3/2} + y^2(x^2+y^2)^{-3/2} \\ &= (x^2+y^2)(x^2+y^2)^{-3/2} \\ &= (x^2+y^2)^{-1/2} \\ &= 1/\sqrt{x^2+y^2} \\ &= 1/r. \end{aligned}$$

That was weirdly complicated, but we got the right answer.

Problem 4. Center of Mass of a 3D Region. Let D be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Think of D as a solid with constant mass density 1. This region can be parametrized by $0 \le x \le 1$, $0 \le y \le 1-x$ and $0 \le z \le 1-x-y$.

(a) Compute the total mass $M = \iiint_D 1\, dxdydz$.

(b) Compute the moments

$$M_x = \iiint_D x\, dxdydz, \quad M_y = \iint_D y\, dxdydz, \quad M_z = \iiint_D z\, dxdydz.$$

[Hint: There might be a shortcut.]

(c) Compute the center of mass.

(a): The total mass (i.e., the volume) is

$$\begin{aligned}
M &= \iiint_D 1\, dxdydz \\
&= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1\, dz \right) dy \right) dx \\
&= \int_0^1 \left(\int_0^{1-x} (1-x-y)\, dy \right) dx \\
&= \int_0^1 \left[y - xy - \frac{1}{2}y^2 \right]_0^{1-x} dx \\
&= \int_0^1 \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx \\
&= \int_0^1 \left[\frac{1}{2}x^2 - x + \frac{1}{2} \right] dx \\
&= \left[\frac{1}{2}\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x \right]_0^1 \\
&= \frac{1}{2}\frac{1}{3} - \frac{1}{2} + \frac{1}{2} \\
&= 1/6.
\end{aligned}$$

(b): Because the shape D is symmetric under permuting x, y, z we know that $M_x = M_y = M_z$. It turns out that M_x is easiest to compute:

$$\begin{aligned}
M_x &= \iiint_D x\, dxdydz \\
&= \int_0^1 x \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1\, dz \right) dy \right) dx \\
&= \int_0^1 x \left(\int_0^{1-x} (1-x-y)\, dy \right) dx \\
&= \int_0^1 x \left[y - xy - \frac{1}{2}y^2 \right]_0^{1-x} dx \\
&= \int_0^1 x \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx \\
&= \int_0^1 x \left[\frac{1}{2}x^2 - x + \frac{1}{2} \right] dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left[\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right] dx \\
&= \left[\frac{1}{2}\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}\frac{1}{2}x^2\right]_0^1 \\
&= \frac{1}{2}\frac{1}{4} - \frac{1}{3} + \frac{1}{2}\frac{1}{2} \\
&= 1/24.
\end{aligned}
$$

(c): The center of mass is

$$
(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_x}{M}, \frac{M_y}{M}, \frac{M_z}{M}\right) = \left(\frac{1/24}{1/6}, \frac{1/24}{1/6}, \frac{1/24}{1/6}\right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).
$$

Remark: Consider the solid n-dimensional "simplex" with $n+1$ vertices:

$$
(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1).
$$

Using the same method, one can show that the n-dimensional "hypervolume" is $1/n!$ and the center of mass is $(1, 1, \ldots, 1)/(n+1)$. However, as you can imagine, the computation is messy.

Problem 5. Cylindrical Coordinates. Let D be a solid cone of radius 1 and height 1. We can think of this as the solid region defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1 - \sqrt{x^2 + y^2}$. Use cylindrical coordinates to compute the integral

$$
\iiint_D z \, dxdydz.
$$

[Hint: Cylindrical coordinates are defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and satisfy $\partial(x, y, z)/\partial(r, \theta, z) = r$. That is, $dxdydz = r\,drd\theta dz$.]

In cylindrical coordinates, the cone D has parametrization $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1 - r$. Hence

$$
\begin{aligned}
\iiint_D z\,drd\theta dz &= \iiint_D zr\,drd\theta dz \\
&= 2\pi \int_0^1 r \left(\int_0^{1-r} z\,dz\right) dr \\
&= 2\pi \int_0^1 r \left[\frac{1}{2}z^2\right]_0^{1-r} dr \\
&= 2\pi \int_0^1 r(1-r)^2 dr \\
&= \pi \int_0^1 \left[r^3 - 2r^2 + r\right] dr \\
&= \pi \left[\frac{1}{4}r^4 - 2\frac{1}{3}r^3 + \frac{1}{2}r^2\right]_0^1 \\
&\pi \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) \\
&= \pi/12.
\end{aligned}
$$

Remark: If the cone has uniform density 1 then we just computed M_z. The volume of a cone is $(1/3)\pi(\text{radius})^2(\text{height}) = \pi/3$ and by symmetry we have $M_x = M_y = 0$, hence the center of mass of the cone is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_x}{M}, \frac{M_y}{M}, \frac{M_z}{M}\right) = \left(0, 0, \frac{\pi/12}{\pi/3}\right) = \left(0, 0, \frac{1}{4}\right).$$

That is, the center of mass on the main axis at 1/4 of the height. This same result holds for any cone of any radius and height.

Problem 6. Spherical coordinates ρ, ϕ, θ are defined by

$$\begin{aligned} x &= \rho \sin\phi \cos\theta, \\ y &= \rho \sin\phi \sin\theta, \\ z &= \rho \cos\phi, \end{aligned}$$

and satisfy $\partial(x, y, z)/\partial(r, \phi, \theta) = \rho^2 \sin\phi$. That is, $dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta$. Use spherical coordinates to compute the integral

$$\iiint_D \frac{1}{x^2 + y^2 + z^2} \, dxdydz,$$

where D is the unit sphere. Even though the function $f(x, y, z) = 1/(x^2 + y^2 + z^2)$ goes to infinity when $(x, y, z) \to (0, 0, 0)$, the integral is still finite.

$$\begin{aligned} \iiint_D \frac{1}{x^2 + y^2 + z^2} \, dxdydz &= \iiint_D \frac{1}{\rho^2} \rho^2 \sin\phi \, d\rho d\phi d\theta \\ &= \iiint_D \sin\phi \, d\rho d\phi d\theta \\ &= \int_0^{2\pi} 1 \, d\theta \int_0^{\rho} 1 \, d\rho \int_0^{\pi} \sin\phi \, d\phi \\ &= 2\pi \left[-\cos\phi\right]_0^{\pi} \\ &= 2\pi \left[-\cos(\pi) + \cos(0)\right] \\ &= 2\pi \left[-(-1) + (1)\right] \\ &= 4\pi. \end{aligned}$$

Remark: This looked like the hardest problem on HW4, but it was actually the easiest!

Remark: The analogous integral in one dimension is $\int_{-1}^{1} (1/x^2) \, dx$, which diverges. The analogous integral in two dimensions also diverges:

$$\begin{aligned} \iint_{\text{unit disk}} \frac{1}{x^2 + y^2} \, dxdy &= \iint \frac{1}{r^2} \, r \, dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 \frac{1}{r} \, dr \\ &= 2\pi \left[\ln(1) - \ln(0)\right] \\ &= 2\pi \left[0 - (-\infty)\right] \\ &= \infty. \end{aligned}$$

For some reason the three dimensional version converges. We will observe the same type of phenomenon when we study gravity.