Problem 1. Area of a Parametrized Region. Given a region D in $\mathbb{R}^2$, the area is

$$\text{Area}(D) = \iint_D 1\,dxdy.$$

For each of the following problems you should (1) draw the region, (2) find a parametrization, (3) use your parametrization to compute the area.

(a) The half-circle satisfying $x^2 + y^2 \leq 4$ and $x \geq 0$. [Hint: Use polar coordinates.]
(b) The region satisfying $x^2 + y^2 \leq 4$ and $x \geq 1$. [Hint: Don't use polar coordinates. You will need the antiderivative

$$\int 2\sqrt{4-x^2}\,dx = x\sqrt{4-x^2} + 4\arcsin(x/2).]$$

Problem 2. Center of Mass of a 2D Region. Let D be the region parametrized by $0 \leq x \leq 2$ and $x \leq y \leq 5x - 2x^2$. Think of D as a solid with mass density 1.

(a) Compute the total mass $M = \iint_D 1\,dxdy$.
(b) Compute the moments $M_x = \iint_D x\,dxdy$ and $M_y = \iint_D y\,dxdy$.
(c) Compute the center of mass.
(d) Draw the region and its center of mass.

Problem 3. Polar Coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. We already know that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = r.$$

The general theory predicts that we must also have

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \frac{1}{r}.$$

Check that this is true. [Hint: $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$.]

Problem 4. Center of Mass of a 3D Region. Let D be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Think of D as a solid with constant mass density 1. This region can be parametrized by $0 \leq x \leq 1$, $0 \leq y \leq 1-x$ and $0 \leq z \leq 1-x-y$.

(a) Compute the total mass $M = \iiint_D 1\,dxdydz$.
(b) Compute the moments

$$M_x = \iiint_D x\,dxdydz, \quad M_y = \iint_D y\,dxdydz, \quad M_z = \iiint_D z\,dxdydz.$$

[Hint: There might be a shortcut.]

(c) Compute the center of mass.

Problem 5. Cylindrical Coordinates. Let D be a solid cone of radius 1 and height 1. We can think of this as the solid region defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1 - \sqrt{x^2+y^2}$. Use cylindrical coordinates to compute the integral

$$\iiint_D z\,dxdydz.$$

[Hint: Cylindrical coordinates are defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and satisfy $\partial(x,y,z)/\partial(r,\theta,z) = r$. That is, $dxdydz = r\,drd\theta dz$.]

Problem 6. Spherical coordinates ρ, ϕ, θ are defined by

$$\begin{aligned} x &= \rho\sin\phi\cos\theta, \\ y &= \rho\sin\phi\sin\theta, \\ z &= \rho\cos\phi, \end{aligned}$$

and satisfy $\partial(x,y,z)/\partial(r,\phi,\theta) = \rho^2\sin\phi$. That is, $dxdydz = \rho^2\sin\phi\,d\rho d\phi d\theta$. Use spherical coordinates to compute the integral

$$\iiint_D \frac{1}{x^2+y^2+z^2}\,dxdydz,$$

where D is the unit sphere. Even though the function $f(x,y,z) = 1/(x^2+y^2+z^2)$ goes to infinity when $(x,y,z) \to (0,0,0)$, the integral is still finite.