Math 310 **Spring 2024**
Homework 3 **Drew Armstrong**

Problem 1. Tangent Lines to Implicit Curves. Consider a curve of the form $f(x,y)=0$ for some function $f:\mathbb{R}^2\to\mathbb{R}$. Let (x_0,y_0) be some point on the curve, so that $f(x_0,y_0)=0$. Then the tangent line to this curve at the point (x_0,y_0) has equation

$$\nabla f(x_0,y_0)\bullet\langle x-x_0,y-y_0\rangle=0$$
$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)=0.$$

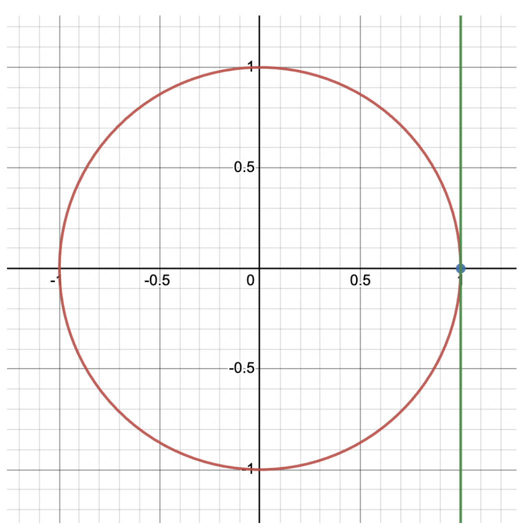
Find the equation of the tangent line in the following situations. In each case, use a computer (e.g., `desmos.com`) to sketch the curve and the line:

(a) $f(x,y)=x^2+y^2-1$ and $(x_0,y_0)=(1,0)$
(b) $f(x,y)=x^2+3y^2-1$ and $(x_0,y_0)=(1/2,1/2)$
(c) $f(x,y)=x^3+x^2-y^2$ and $(x_0,y_0)=(3,6)$
(d) Try $f(x,y)=x^3+x^2-y^2$ and $(x_0,y_0)=(0,0)$. Observe that something goes wrong.

(a): The gradient field is $\nabla f(x,y)=\langle f_x,f_y\rangle=\langle 2x,2y\rangle$. Hence the equation of the tangent line at the point $(x_0,y_0)=(1,0)$ is

$$\begin{aligned}\nabla f(1,0)\bullet\langle x-1,y-0\rangle&=0\\ \langle 2(1),2(0)\rangle\bullet\langle x-1,y-0\rangle&=0\\ \langle 2,0\rangle\bullet\langle x-1,y-0\rangle&=0\\ 2(x-1)+0(y-0)&=0\\ 2(x-1)&=0\\ x-1&=0\\ x&=1.\end{aligned}$$

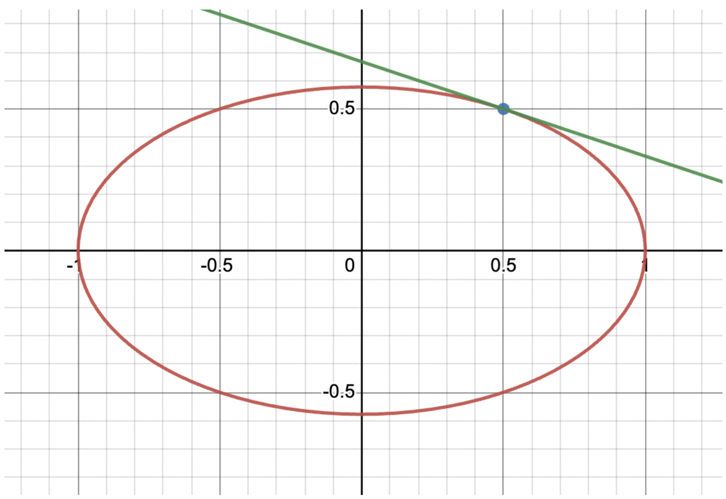
Here is a picture:



(b): The gradient field is $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle 2x, 6y\rangle$. Hence the equation of the tangent line at the point $(x_0, y_0) = (1/2, 1/2)$ is

$$\begin{aligned}
\nabla f(1/2,1/2) \bullet \langle x-1/2, y-1/2\rangle &= 0\\
\langle 2(1/2), 6(1/2)\rangle \bullet \langle x-1/2, y-1/2\rangle &= 0\\
\langle 1,3\rangle \bullet \langle x-1/2, y-1/2\rangle &= 0\\
1(x-1/2)+3(y-1/2) &= 0\\
x+3y-1/2-3/2 &= 0\\
x+3y-2 &= 0\\
x+3y &= 2.
\end{aligned}$$

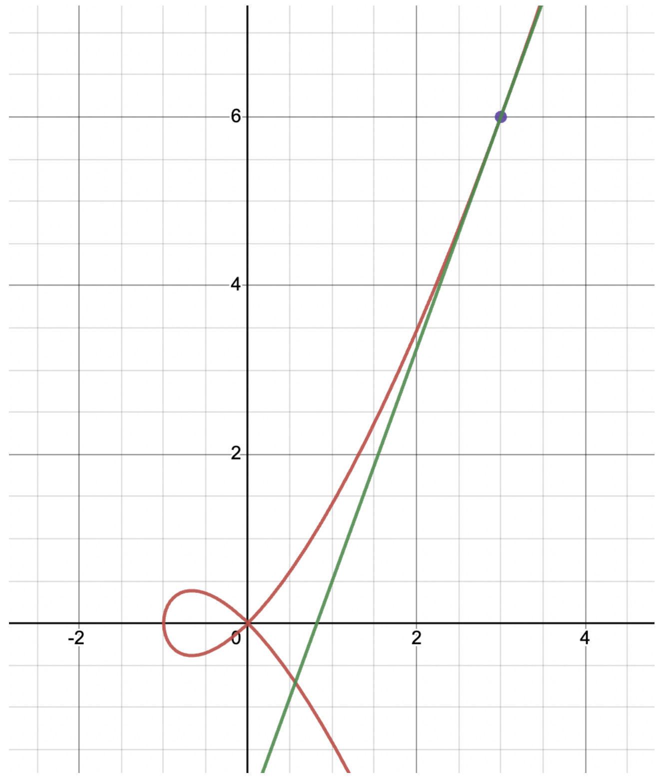
Here is a picture:



(c): The gradient field is $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle 3x^2+2x, -2y\rangle$. Hence the equation of the tangent line at the point $(x_0, y_0) = (3, 6)$ is

$$\begin{aligned}
\nabla f(3,6) \bullet \langle x-3, y-6\rangle &= 0\\
\langle 3(3)^2+2(3), -2(6)\rangle \bullet \langle x-3, y-6\rangle &= 0\\
\langle 33,-12\rangle \bullet \langle x-3, y-6\rangle &= 0\\
33(x-3)-12(y-6) &= 0\\
33x-12y-99+72 &= 0\\
33x-12y-27 &= 0\\
33x-12y &= 27\\
11x-4y &= 9.
\end{aligned}$$

Here is a picture:



Problem 2. Tangent Plane to an Ellipsoid. A function $f : \mathbb{R}^3 \to \mathbb{R}$ defines an implicit surface $f(x, y, z) = 0$. If $f(x_0, y_0, z_0) = 0$ then the tangent plane to this surface at the point (x_0, y_0, z_0) has equation

$$\nabla f(x_0, y_0, z_0) \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$
$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0.$$

Suppose that (x_0, y_0, z_0) is some fixed point on the ellipsoid $ax^2 + by^2 + cz^2 = 1$. Use the above formula to show that the tangent plane to the ellipsoid at (x_0, y_0, z_0) has equation

$$ax_0x + by_0y + cz_0z = 1.$$

[Hint: There is a nice simplification.]

The surface $ax^2 + by^2 + cz^2 = 1$ can be expressed as the level surface $f(x, y, z) = 0$ where $f(x, y, z) = ax^2 + by^2 + cz^2 - 1$. The gradient field is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x, f_y, f_z \rangle \\ &= \langle 2ax, 2by, 2cz \rangle.\end{aligned}$$

Let (x_0, y_0, z_0) be any point on the surface $ax^2 + by^2 + cz^2 = 1$, so that $ax_0^2 + by_0^2 + cz_0^2 = 1$. Then the equation of the tangent plane to the surface at this point is

$$\begin{aligned}\nabla f(x_0, y_0, z_0) \bullet \langle x - x_0, y - y_0, z - z_0\rangle &= 0\\ \langle 2ax_0, 2by_0, 2cz_0\rangle \bullet \langle x - x_0, y - y_0, z - z_0\rangle &= 0\\ 2ax_0(x - x_0) + 2by_0(y - y_0) + 2cz_0(z - z_0) &= 1\\ 2axx_0 + 2byy_0 + 2czz_0 - 2ax_0^2 - 2by_0^2 - 2cz_0^2 &= 0\\ 2axx_0 + 2byy_0 + 2czz_0 &= 2ax_0^2 + 2by_0^2 + 2cz_0^2\\ axx_0 + byy_0 + czz_0 &= ax_0^2 + by_0^2 + cz_0^2\\ axx_0 + byy_0 + czz_0 &= 1.\end{aligned}$$

Isn't that nice?

Problem 3. The Multivariable Chain Rule. Let $f(x, y, z)$ be a function of x, y, z and let $x(t), y(t), z(t)$ be functions of t, so $f(t) = f(x(t), y(t), z(t))$ is also a function of t. The multivariable chain rule says that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Equivalently, if we think of $\mathbf{r}(t) = (x(t), y(t), z(t))$ as a parametrized path, then we can express the chain in terms of the gradient vector and the dot product:

$$[f(\mathbf{r}(t))]' = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t).$$

(a) Compute df/dt when $f(x, y) = xy$, $x(t) = \cos t$ and $y(t) = \sin t$.
(b) Suppose that a path $\mathbf{r}(t)$ satisfies $f(\mathbf{r}(t)) = 7$ for all t. In this case, prove that the velocity $\mathbf{r}'(t)$ is perpendicular to the gradient vector $\nabla f(\mathbf{r}(t))$ at the point $\mathbf{r}(t)$.

(a): If $f(x, y) = xy$, $x(t) = \cos t$ and $\mathbf{y}(t) = \sin t$ then we have

$$\begin{aligned}\partial f/\partial x &= y,\\ \partial f/\partial y &= x,\\ dx/dt &= -\sin t,\\ dy/dt &= \cos t,\end{aligned}$$

and hence

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= (y)(-\sin t) + (x)(\cos t)\\ &= (\sin t)(-\sin t) + (\cos t)(\cos t)\\ &= \cos^2 t - \sin^2 t\\ &= \cos(2t).\end{aligned}$$

This can also be computed **without** the chain rule. First substitute $x(t)$ and $y(t)$ into $f(x, y)$:

$$f(t) = f(x(t), y(t)) = f(\cos t, \sin t) = (\cos t)(\sin t).$$

Then differentiate using the product rule:

$$\begin{aligned} f'(t) &= (\cos t)(\sin t)' + (\cos t)'(\sin t) \\ &= (\cos t)(\cos t) + (-\sin t)(\sin t) \\ &= \cos^2 t - \sin^2 t. \end{aligned}$$

(b): Let $f : \mathbb{R}^n \to \mathbb{R}$ be any scalar function and let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ be any path satisfying $f(\mathbf{r}(t)) = 7$ for all t. Then by the chain rule we have

$$\begin{aligned} [f(\mathbf{r}(t))]' &= [7]' \\ \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) &= 0, \end{aligned}$$

which tells us that the vectors $\nabla f(\mathbf{r}(t))$ and $\mathbf{r}'(t)$ are perpendicular for all t. Geometric meaning: The particle is traveling within the level surface $f = 7$, so the velocity vector $\mathbf{r}'(t)$ is tangent to this surface at the point $\mathbf{r}(t)$. Since $\nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = 0$ we conclude that the gradient vector $\nabla f(\mathbf{r}(t))$ is perpendicular to the level surface at the point $\mathbf{r}(t)$. This is the most important fact about gradient vectors.

Problem 4. Gradient Flow. Let $f(x, y, z)$ denote the concentration of krill at point (x, y, z) in the ocean. Suppose you are a whale swimming with trajectory $\mathbf{r}(t)$ and suppose that your speed is constant, say $\|\mathbf{r}'(t)\| = 1$.

(a) According to the multivariable chain rule, the rate of change of krill near you is $[f(\mathbf{r}(t))]' = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)$. Explain why this rate of change is maximized when your velocity is parallel to the gradient vector $\nabla(\mathbf{f}(t))$. [Hint: Use the dot product theorem $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$.]

(b) For a simple example, take $f(x, y, z) = x^2 + xy + y^2 - z^2$. And suppose your current position is $(1, 1, 1)$. In which direction should you swim in order to maximize your intake of krill?

(a): If the whale travels at constant speed then we have $\|\mathbf{r}'(t)\| = 1$. The concentration of krill at the whale's position is $f(\mathbf{r}'(t))$, hence the rate of change of concentration is

$$[f(\mathbf{r}(t))]' = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = \|\nabla f(\mathbf{r}(t))\|\|\mathbf{r}'(t)\| \cos\theta = \|\nabla f(\mathbf{r}(t))\| \cos\theta,$$

where θ is the angle between the whale's velocity $\mathbf{r}'(t)$ and the krill gradient $\nabla f(\mathbf{r}(t))$ at the whale's position. This quantity is maximized when $\theta = 0$, i.e., when the whale is swimming parallel to the gradient.

(b): If $f(x, y, z) = x^2 + xy + y^2 - z^2$ then we have

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle = \langle 2x + y, x + y^2, -2z \rangle.$$

If the whale is currently at position $\mathbf{r}(t) = (1, 1, 1)$ (the value of t is not important) then in order to maximize the intake of krill the whale should swim in the direction of the gradient vector

$$\nabla f(1, 1, 1) = \langle 2(1) + (1), (1) + (1)^2, -2(1) \rangle = \langle 3, 3, -2 \rangle.$$

Problem 5. Linear Approximation. The multivariable chain rule can also be expressed in terms of "linear approximation". Consider a function $f(x, y)$. If the inputs change by small amounts Δx and Δy, then the out put changes by a small amount Δf, which satisfies the following approximation:

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

Now consider a cylinder with radius r and height h. Suppose that you measure the radius and the height to be approximately $r = 10$ cm and $h = 15$ cm, so the volume of the cylinder is approximately $V = \pi r^2 h = \pi(10)^2(15) = 1500\pi$ cm^2.

(a) If your ruler has a sensitivity of 0.1 cm, estimate the error in the computed value of V. [Hint: Let $\Delta r = 0.1$ and $\Delta h = 0.1$. You want to estimate ΔV.]
(b) Find the percent errors in r, h and V. What do you notice?

(a): The volume of the cylinder is $V = \pi r^2 h$, which is a function of r and h. The linear approximation formula tells us that

$$\begin{aligned}\Delta V &\approx V_r \Delta r + V_h \Delta h \\ &= 2\pi r h \Delta r + \pi r^2 \Delta h \\ &= \pi(2rh\Delta r + r^2 \Delta h).\end{aligned}$$

If we measure $r = 10$ and $h = 15$ then our calculated value of V is $\pi(10)^2(15) = 1500\pi$ cm^2. If our ruler has sensitivity 0.1 cm then the errors in r and h are $\Delta r = \Delta h = 0.1$. Hence the approximate error in our calculated value of V is

$$\Delta V \approx \pi(2(10)(15)(0.1) + (10)^2(0.1)) = 40\pi.$$

The percent errors in r and h are $\Delta r/r = 0.1/10 = 1\%$ and $\delta h/h = 0.1/15 = 0.67\%$. The percent error in our calculated value of V is $\Delta V/V = 40\pi/1500\pi = 40/1500 = 2.66\%$. Note that the percent error of the output is larger than the percent error of the input.

Problem 6. Multivariable Optimization. Consider the scalar field $f(x, y) = x^3 + xy - y^3$.

(a) Compute the gradient vector field $\nabla f(x, y)$.
(b) Find all critical points; i.e., points (a, b) such that $\nabla f(a, b) = \langle 0, 0\rangle$.
(c) Compute the Hessian matrix $Hf(x, y)$ and its determinant.
(d) Use the "second derivative test" to determine whether each of the critical points from part (b) is a local maximum, local minimum, or a saddle point.

(a): The gradient vector is

$$\nabla f = \langle f_x, f_y\rangle = \langle 3x^2 + y, x - 3y^2\rangle.$$

(b): To find the critical points we must solve the following system of nonlinear equations:

$$\begin{cases} 3x^2 + y &= 0, \\ x - 3y^2 &= 0. \end{cases}$$

Solving the second equation for x gives $x = 3y^2$ then substituting into the first equation gives

$$\begin{aligned} 3x^2 + y &= 0 \\ 3(3y^2)^2 + y &= 0 \\ 27y^4 + y &= 0 \\ y(27y^3 + 1) &= 0. \end{aligned}$$

This implies that $y = 0$ or

$$\begin{aligned} 27y^3 + 1 &= 0 \\ 27y^3 &= -1 \\ y^3 &= -1/27 \\ y &= -1/3. \end{aligned}$$

(Recall that a negative real number has a unique real cube root.) When $y = 0$ we must have $x = 3y^2 = 0$ and when $y = -1/3$ we must have $x = 3y^2 = 3(-1/3)^2 = 1/3$. Hence there are exactly two critical points: $(0, 0)$ and $(1/3, -1/3)$.

(c): To compute the determinant of the Hessian matrix we must first compute all second derivatives of f:

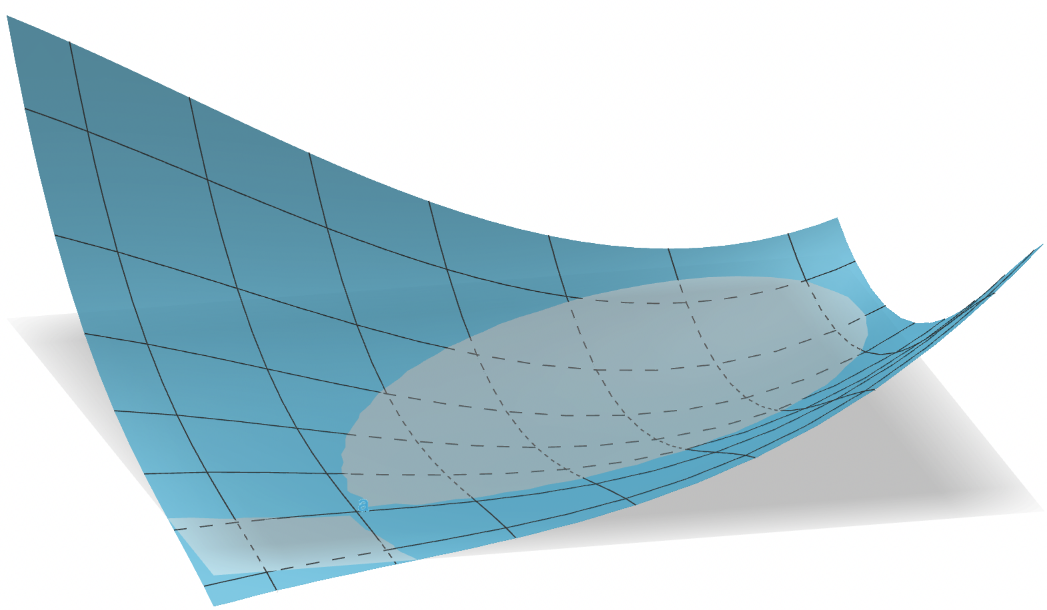
$$\begin{aligned} f_{xx} &= 6x, \\ f_{yy} &= -6y, \\ f_{xy} &= 1, \\ f_{yx} &= 1. \end{aligned}$$

Thus the Hessian determinant is

$$\det(Hf) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det \begin{pmatrix} 6x & 1 \\ 1 & -6y \end{pmatrix} = -36xy - 1.$$

Since $\det(Hf)(0, 0) = -1 < 0$ we see that $(0, 0)$ is a saddle point. Since $\det(Hf)(1/3, -1/3) = -36(-1/9) - 1 = 3 > 0$ we see that $(1/3, -1/3)$ is a local maximum or minimum. Since $f_{xx}(1/3, -1/3) = 6(1/3) = 2 > 0$ we see that $(1/3, -1/3)$ is a local minimum.[1]

Here is a picture:



[1]We could also check that $f_{yy}(1/3, -1/3) = -6(-1/3) = 2 > 0$.