Math 310 **Spring 2024**
Homework 3 **Drew Armstrong**

Problem 1. Tangent Lines to Implicit Curves. Consider a curve of the form $f(x,y)=0$ for some function $f:\mathbb{R}^2\to\mathbb{R}$. Let (x_0,y_0) be some point on the curve, so that $f(x_0,y_0)=0$. Then the tangent line to this curve at the point (x_0,y_0) has equation

$$\nabla f(x_0,y_0)\bullet\langle x-x_0,y-y_0\rangle=0$$
$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)=0.$$

Find the equation of the tangent line in the following situations. In each case, use a computer (e.g., `desmos.com`) to sketch the curve and the line:

(a) $f(x,y)=x^2+y^2-1$ and $(x_0,y_0)=(1,0)$
(b) $f(x,y)=x^2+3y^2-1$ and $(x_0,y_0)=(1/2,1/2)$
(c) $f(x,y)=x^3+x^2-y^2$ and $(x_0,y_0)=(3,6)$
(d) Try $f(x,y)=x^3+x^2-y^2$ and $(x_0,y_0)=(0,0)$. Observe that something goes wrong.

Problem 2. Tangent Plane to an Ellipsoid. A function $f:\mathbb{R}^3\to\mathbb{R}$ defines an implicit surface $f(x,y,z)=0$. If $f(x_0,y_0,z_0)=0$ then the tangent plane to this surface at the point (x_0,y_0,z_0) has equation

$$\nabla f(x_0,y_0,z_0)\bullet\langle x-x_0,y-y_0,z-z_0\rangle=0$$
$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0)+\frac{\partial f}{\partial z}(z-z_0)=0.$$

Suppose that (x_0,y_0,z_0) is some fixed point on the ellipsoid $ax^2+by^2+cz^2=1$. Use the above formula to show that the tangent plane to the ellipsoid at (x_0,y_0,z_0) has equation

$$ax_0x+by_0y+cz_0z=1.$$

[Hint: There is a nice simplification.]

Problem 3. The Multivariable Chain Rule. Let $f(x,y,z)$ be a function of x,y,z and let $x(t),y(t),z(t)$ be functions of t, so $f(t)=f(x(t),y(t),z(t))$ is also a function of t. The multivariable chain rule says that

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Equivalently, if we think of $\mathbf{r}(t)=(x(t),y(t),z(t))$ as a parametrized path, then we can express the chain in terms of the gradient vector and the dot product:

$$[f(\mathbf{r}(t))]'=\nabla f(\mathbf{r}(t))\bullet\mathbf{r}'(t).$$

(a) Compute df/dt when $f(x,y)=xy$, $x(t)=\cos t$ and $y(t)=\sin t$.
(b) Suppose that a path $\mathbf{r}(t)$ satisfies $f(\mathbf{r}(t))=7$ for all t. In this case, prove that the velocity $\mathbf{r}'(t)$ is perpendicular to the gradient vector $\nabla f(\mathbf{r}(t))$ at the point $\mathbf{r}(t)$.

Problem 4. Gradient Flow. Let $f(x,y,z)$ denote the concentration of algae at point (x,y,z) in the ocean. Suppose you are a whale swimming with trajectory $\mathbf{r}(t)$ and suppose that your speed is constant, say $\|\mathbf{r}'(t)\|=1$.

(a) According to the multivariable chain rule, the rate of change of algae near you is $[f(\mathbf{r}(t))]' = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)$. Explain why this rate of change is maximized when your velocity is parallel to the gradient vector $\nabla(\mathbf{f}(t))$. [Hint: Use the dot product theorem $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$.]

(b) For a simple example, take $f(x, y, z) = x^2 + xy + y^2 - z^2$. And suppose your current position is $(1, 1, 1)$. In which direction should you swim in order to maximize your intake of algae?

Problem 5. Linear Approximation. The multivariable chain rule can also be expressed in terms of "linear approximation". Consider a function $f(x, y)$. If the inputs change by small amounts Δx and Δy, then the out put changes by a small amount Δf, which satisfies the following approximation:

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

Now consider a cylinder with radius r and height h. Suppose that you measure the radius and the height to be approximately $r = 10$ cm and $h = 15$ cm, so the volume of the cylinder is approximately $V = \pi r^2 h = \pi(10)^2(15) = 1500\pi$ cm^2.

(a) If your ruler has a sensitivity of 0.1 cm, estimate the error in the computed value of V. [Hint: Let $\Delta r = 0.1$ and $\Delta h = 0.1$. You want to estimate ΔV.]

(b) Find the percent errors in r, h and V. What do you notice?

Problem 6. Multivariable Optimization. Consider the scalar field $f(x, y) = x^3 + xy - y^3$.

(a) Compute the gradient vector field $\nabla f(x, y)$.

(b) Find all critical points, i.e., points (a, b) such that $\nabla f(a, b) = \langle 0, 0 \rangle$.

(c) Compute the Hessian matrix $Hf(x, y)$ and its determinant.

(d) Use the "second derivative test" to determine whether each of the critical points from part (b) is a local maximum, local minimum, or a saddle point.