Problem 1. Describing Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$. In each part, there are infinitely many correct ways to express the answer.

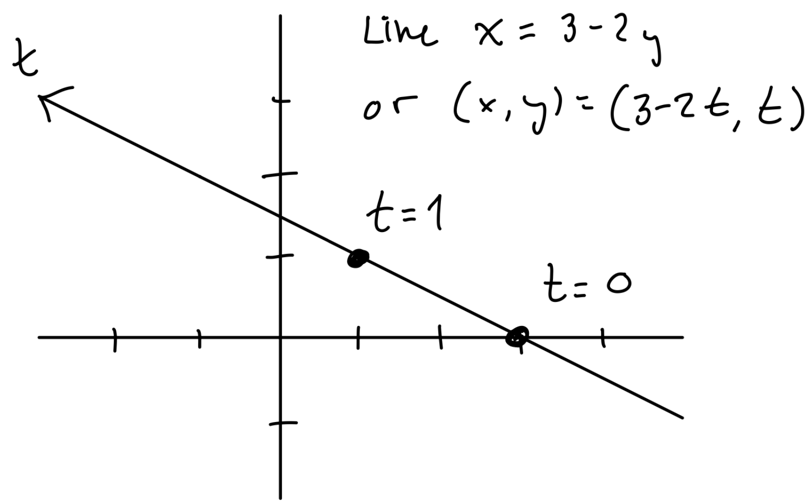
(a) Find a parametrization for the line in $\mathbb{R}^2$ passing through points $(3,0)$ and $(1,1)$.
(b) Eliminate the parameter to find the equation of the line from part (a).
(c) Find a parametrization for the line in $\mathbb{R}^3$ passing through points $(1,-1,1)$ and $(0,3,3)$.
(d) Eliminate the parameter to find the **equations of two planes** in $\mathbb{R}^3$ whose intersection is the line from part (c).

(a): The general form of a parametrized line is $\mathbf{r}(t) = \langle x_0 + tu, y_0 + tv\rangle$. This line passes through the point (x_0, y_0) at time $t = 0$ and has constant velocity $\mathbf{r}'(t) = \langle u, v\rangle$. In our case we can take $(x_0, y_0) = (3,0)$ and $\langle u, v\rangle = (1,1) - (3,0) = (-2,1)$ as the velocity vector, to get

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle 3 + t(-2), 0 + t(1)\rangle = \langle 3 - 2t, t\rangle$$

There are infinitely many correct ways to express this.

(b): From part (a) we have $x(t) = 3 - 2t$ and $y(t) = t$. Substituting the second equation into the first gives $x = 3 - 2y$. This is the equation of the line. Picture:



(c): To parametrize a line in $\mathbb{R}^3$ we need a point (x_0, y_0, z_0) and a velocity vector $\langle u, v, w\rangle$. If this line goes through the points $(1,-1,1)$ and $(0,3,3)$ then we can take $(x_0, y_0, z_0) = (1,-1,1)$ and $\langle u, v, w\rangle = (0,3,3) - (1,-1,1) = \langle -1, 4, 2\rangle$. Hence our line is

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle x_0 + tu, y_0 + tv, z_0 + tw\rangle = \langle 1 - t, -1 + 4t, 1 + 2t\rangle.$$

(d): From part (c) we have $x = 1 - t$, $y = -1 + 4t$ and $z = 1 + 2t$. Each of these equations can be solved for t:

$$t = 1 - x,$$
$$t = (y + 1)/4,$$
$$t = (z - 1)/2.$$

Now we have not one, but **three** equations that don't involve t:

$$\begin{aligned} 1 - x &= (y+1)/4, \\ 1 - x &= (z-1)/2, \\ (y+1)/4 &= (z-1)/2. \end{aligned}$$

Each of these is the equation of a plane, and the three planes intersect at the original line. But you can throw away any of the three, and the intersection of the remaining two planes is still the same line. So pick any two that you want.

Problem 2. A Plane in $\mathbb{R}^3$. The following three points in $\mathbb{R}^3$ determine a plane:

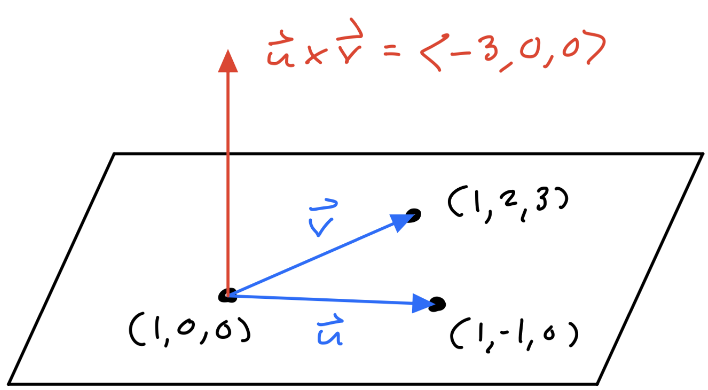
$$P = (1,0,0), \quad Q = (1,-1,0), \quad R = (1,2,3).$$

(a) Find a vector that is perpendicular to this plane. [Hint: Use the cross product.]
(b) Use your answer from part (a) to find the equation of the plane. [Recall: The plane in $\mathbb{R}^3$ that passes through the point (x_0, y_0, z_0) and is perpendicular to the vector $\langle a, b, c\rangle$ has the equation $\langle a, b, c\rangle \bullet \langle x - x_0, y - y_0, z - z_0\rangle = 0$.]

(a): We can find a vector perpendicular to the plane by taking the cross product of any two vectors in the plane. For example, let's take $\mathbf{u} = \vec{PQ} = (1,-1,0) - (1,0,0) = \langle 0,-1,0\rangle$ and $\mathbf{v} = \vec{PR} = (1,2,3) - (1,0,0) = \langle 0,2,3\rangle$, with cross product

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle 0,-1,0\rangle \times \langle 0,2,3\rangle \\ &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix} \\ &= \mathbf{i} \det \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} \\ &= -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\ &= \langle -3, 0, 0\rangle. \end{aligned}$$

Here is a picture:



(b): Let's choose the point $(x_0, y_0, z_0) = (1, 0, 0)$ and the normal vector $\langle u, v, w\rangle = \langle -3, 0, 0\rangle$. Then the equation of the plane is

$$\begin{aligned}\langle a, b, c\rangle \bullet \langle x - x_0, y - y_0, z - z_0\rangle &= 0\\ \langle -3, 0, 0\rangle \bullet \langle x - 1, y - 0, z - 0\rangle &= 0\\ -3(x-1) + 0(y-0) + 0(z-0) &= 0\\ -3(x-1) &= 0\\ x - 1 &= 0\\ x &= 1.\end{aligned}$$

Wow, that's pretty simple.

Problem 3. The Intersection of Two Planes in $\mathbb{R}^3$. The following system of two linear equations in three unknowns represents the intersection of two planes in $\mathbb{R}^3$:

$$\begin{cases} 2x + y - z = 3, \\ x - y + 2z = 1. \end{cases}$$

(a) Express the intersection as a parametrized line in $\mathbb{R}^3$. [Hint: There are many ways to express the answer. I think the easiest way is to let $t = z$ be the parameter. Then solve for x and y in terms of t.]

(b) The two planes have normal vectors $\mathbf{u} = \langle 2, 1, -1\rangle$ and $\mathbf{v} = \langle 1, -1, 2\rangle$. Compute the cross product $\mathbf{u} \times \mathbf{v}$. How is this related to your answer from part (a)?

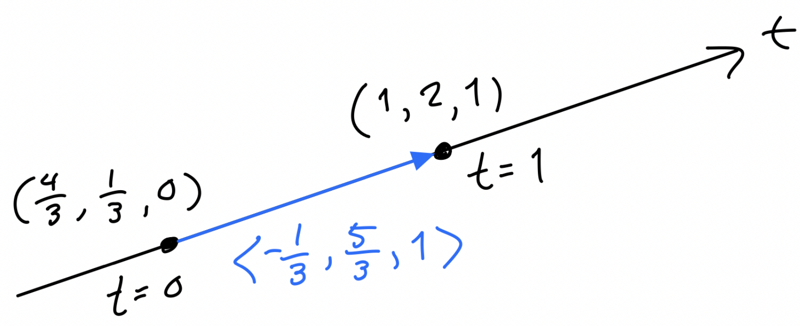
(a): Let's eliminate x first. Subtract twice the second equation from the first to get $3y-5z = 1$. Setting $z = t$ gives $3y - 5t = 1$, of $y = 1/3 + (5/3)t$. Then we back-substitute into the first equation to obtain

$$\begin{aligned}2x + y - z &= 3\\ 2x + \frac{1}{3} + \frac{5}{3}t - t &= 3\\ 2x &= \frac{8}{3} - \frac{2}{3}t\\ x &= \frac{4}{3} - \frac{1}{3}t.\end{aligned}$$

Thus we obtain the parametrized line

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \left\langle \frac{4}{3} - \frac{1}{3}t, \frac{1}{3} + \frac{5}{3}t, t \right\rangle.$$

Note that the velocity of this parametrization is $\mathbf{r}'(t) = \langle -1/3, 5/3, 1\rangle$. Picture:



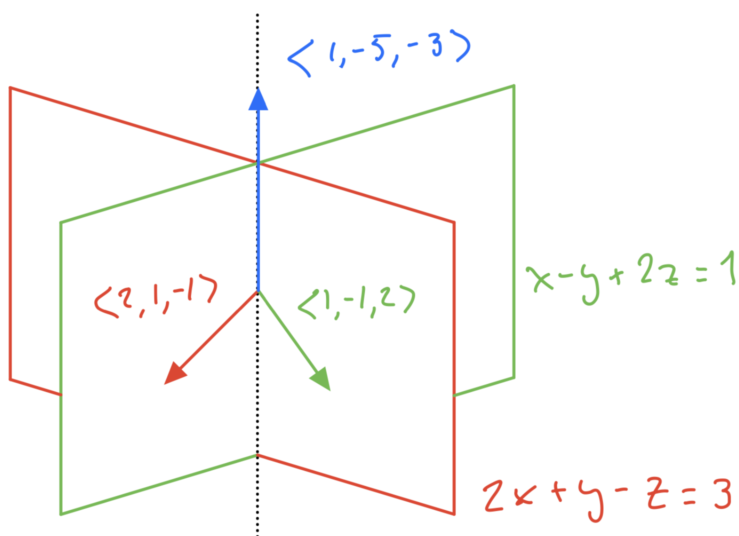
(b): The two planes have normal vectors $\mathbf{u} = \langle 2, 1, -1\rangle$ and $\mathbf{v} = \langle 1, -1, 2\rangle$, with cross product

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \langle 2,1,-1\rangle \times \langle 1,-1,2\rangle \\ &= \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -1\\ 1 & -1 & 2\end{pmatrix} \\ &= \mathbf{i}\det\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix} - \mathbf{j}\det\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix} + \mathbf{k}\det\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix} \\ &= 1\mathbf{i} - 5\mathbf{j} - 3\mathbf{k} \\ &= \langle 1,-5,-3\rangle.\end{aligned}$$

Note that this vector is a scalar multiple of the velocity vector from part (a):

$$\langle 1,-5,-3\rangle = -3\left\langle -\frac{1}{3}, \frac{5}{3}, 1\right\rangle.$$

Since the vector $\mathbf{u}\times\mathbf{v}$ is perpendicular to both normal vectors, it is contained in both planes, hence it is contained in their line of intersection. Picture:



Problem 4. Projectile Motion. A projectile of mass m is launched from the point $(0,0)$ with an initial speed of v, at an angle of θ above the horizontal. Let $\mathbf{r}(t) = (x(t), y(t))$ be the position of the particle at time t.

(a) Neglecting air resistance, Galileo tells us that the acceleration is $\mathbf{r}''(t) = \langle 0, -g\rangle$ for some constant $g > 0$.[1] Use this to find the position $\mathbf{r}(t)$. [Hint: Integrate $\mathbf{r}''(t)$ twice, using the initial conditions $\mathbf{r}(0) = (0,0)$ and $\mathbf{r}'(0) = \langle v\cos\theta, v\sin\theta\rangle$.]

(b) Show that the projectile travels horizontal distance $v^2\sin(2\theta)/g$ before hitting the ground. [Hint: Set $y(t) = 0$ and solve for t.]

(c) Find the value of θ that maximizes the horizontal distance traveled. [Hint: Think of the distance as a function of θ. Compute $d/d\theta$.]

[1] In particular, the acceleration does not depend on the mass m.

(a): Our goal is to find explicit formulas for the position at time t. We begin by integrating $\mathbf{r}''(t) = \langle 0, -g \rangle$ to get $\mathbf{r}'(t)$. Since g is constant we have

$$\begin{aligned}\mathbf{r}'(t) &= \left\langle \int 0\,dt, \int -g\,dt \right\rangle \\ &= \langle c_1, -gt + c_2 \rangle\end{aligned}$$

for come constants of integration c_1, c_2. We use the initial velocity to see that

$$\langle v\cos\theta, v\sin\theta \rangle = \mathbf{r}'(0) = \langle c_1, 0 + c_2 \rangle = \langle c_1, c_2 \rangle,$$

and hence

$$\mathbf{r}'(t) = \langle v\cos\theta, -gt + v\sin\theta \rangle.$$

Next we integrate $\mathbf{r}'(t)$ to get $\mathbf{r}(t)$. Since v, θ and g are constant we have

$$\begin{aligned}\mathbf{r}(t) &= \left\langle \int v\cos\theta\,dt, \int (-gt + v\cos\theta)\,dt \right\rangle \\ &= \left\langle (v\cos\theta)t + c_3, -\frac{1}{2}gt^2 + (v\sin\theta)t + c_4 \right\rangle\end{aligned}$$

for some constants c_3, c_4. We use the initial position to see that

$$\langle 0, 0 \rangle = \mathbf{r}(0) = \langle 0 + c_3, 0 + 0 + c_4 \rangle = \langle c_3, c_4 \rangle,$$

and hence

$$\boxed{\mathbf{r}(t) = \left\langle (v\cos\theta)t, -\frac{1}{2}gt^2 + (v\sin\theta)t \right\rangle.}$$

(b): Before we can find **where** the projectile hits the ground, we first need to know **when** this happens. Thus we need to solve the equation

$$\begin{aligned}y(t) &= 0 \\ -\frac{1}{2}gt^2 + (v\sin\theta)t &= 0 \\ t\left(-\frac{1}{2}gt + v\sin\theta\right) &= 0.\end{aligned}$$

We find that the projectile is on the ground at time $t = 0$ (of course) and also when

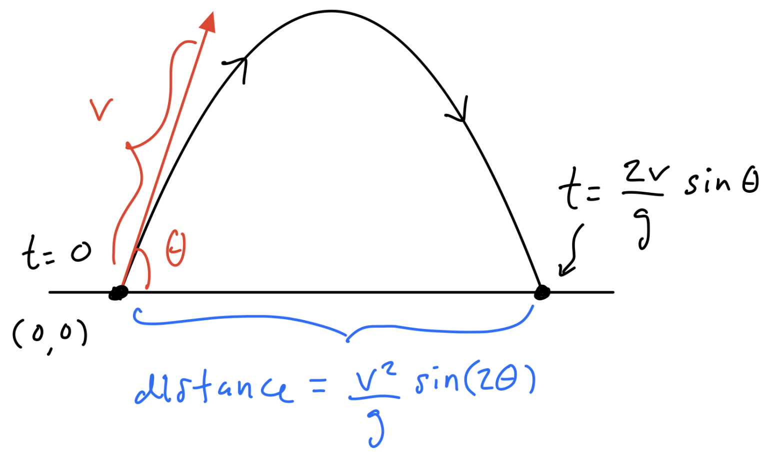
$$\begin{aligned}-\frac{1}{2}gt + v\sin\theta &= 0 \\ t &= \frac{2v}{g}\sin\theta.\end{aligned}$$

Thus the position of the particle when it hits the ground is[2]

$$\begin{aligned}\mathbf{r}\left(\frac{2v}{g}\sin\theta\right) &= \left\langle (v\cos\theta)\frac{2v}{g}\sin\theta, 0 \right\rangle \\ &= \left\langle \frac{2v^2}{g}\sin\theta\cos\theta, 0 \right\rangle \\ &= \left\langle \frac{v^2}{g}\sin(2\theta), 0 \right\rangle.\end{aligned}$$

Here is a picture:

[2]Here I use the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to make the following computations simpler.



(c): For which value of θ is the distance $v^2 \sin(2\theta)/g$ maximized? To solve this we will think of the distance as a function of θ, with v and g fixed:

$$f(\theta) = \frac{v^2}{g}\sin(2\theta).$$

Then to maximize $f(\theta)$ we take the derivative with respect to θ and set this equal to zero:

$$\begin{aligned} df/d\theta &= 0 \\ \frac{v^2}{g}\cos(2\theta)\cdot 2 &= 0 \\ \cos(2\theta) &= 0. \end{aligned}$$

We conclude that $2\theta = 90°$ and hence $\theta = 45°$. Summary: The horizontal distance of a cannonball is maximized by aiming the cannon at $45°$ above the horizontal. This is true on any planet and for any initial speed.

Problem 5. Derivatives of Dot Products and Cross Products. Let $\mathbf{f}, \mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ be any[3] two parametrized paths in $\mathbb{R}^3$. Then we have the following "product rules":

$$\begin{aligned} [\mathbf{f}(t) \bullet \mathbf{g}(t)]' &= \mathbf{f}(t) \bullet \mathbf{g}'(t) + \mathbf{f}'(t) \bullet \mathbf{g}(t), \\ [\mathbf{f}(t) \times \mathbf{g}(t)]' &= \mathbf{f}(t) \times \mathbf{g}'(t) + \mathbf{f}'(t) \times \mathbf{g}(t). \end{aligned}$$

(a) Let $\mathbf{r}(t)$ be the trajectory of a particle traveling on the surface of a sphere centered at $(0,0,0)$. In this case, prove that $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all t. [Hint: By assumption we have $\|\mathbf{r}(t)\| = c$ for some constant c independent of t. Use the fact that $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \bullet \mathbf{r}(t)$.]

(b) Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be the trajectory of a particle in space, and suppose that we have $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar function $c : \mathbb{R} \to \mathbb{R}$. In this case, prove that

$$[\mathbf{r}(t) \times \mathbf{r}'(t)]' = \langle 0, 0, 0\rangle \text{ for all } t.$$

[Hint: Recall that we have $\mathbf{v} \times \mathbf{v} = \langle 0,0,0\rangle$ for any vector $\mathbf{v} \in \mathbb{R}^3$.]

[3] We assume that the derivatives $\mathbf{f}'(t)$ and $\mathbf{g}'(t)$ exist.

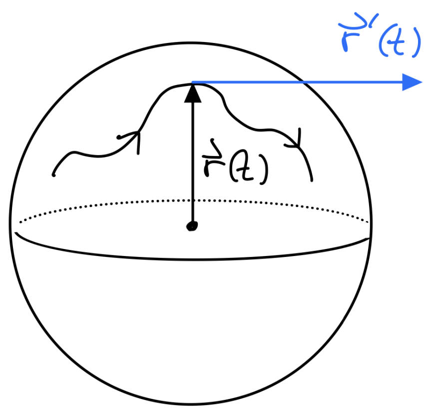
(a): If a particle travels on a sphere of radius c centered at $(0,0,0)$ then we must have $\|\mathbf{r}(t)\| = c$ for all t. Since $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \bullet \mathbf{r}(t)$ we must also have

$$\begin{aligned} \|\mathbf{r}(t)\| &= c \\ \|\mathbf{r}(t)\|^2 &= c^2 \\ \mathbf{r}(t) \bullet \mathbf{r}(t) &= c^2. \end{aligned}$$

Now we take the derivative of both sides and apply the product rule:

$$\begin{aligned} [\mathbf{r}(t) \bullet \mathbf{r}(t)]' &= [c^2]' \\ \mathbf{r}'(t) \bullet \mathbf{r}(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 && c^2 \text{ is constant} \\ \mathbf{r}(t) \bullet \mathbf{r}'(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 \\ 2\mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 \\ \mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0. \end{aligned}$$

In other words, the velocity of the particle is always tangent to the sphere. Here is a picture:



(b): Let $\mathbf{r}(t)$ be the trajectory of a particle in $\mathbb{R}^3$ and assume that the acceleration and position vectors are in the same direction, i.e., that $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar function $c(t)$. Then by using the product rule for the derivative of a cross product we find that

$$\begin{aligned} [\mathbf{r}(t) \times \mathbf{r}'(t)]' &= \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) \\ &= \langle 0,0,0\rangle + \mathbf{r}(t) \times [c(t)\mathbf{r}(t)] \\ (*) \qquad &= \langle 0,0,0\rangle + c(t)[\mathbf{r}(t) \times \mathbf{r}(t)] \\ &= \langle 0,0,0\rangle + c(t)\langle 0,0,0\rangle \\ &= \langle 0,0,0\rangle. \end{aligned}$$

(In step $(*)$ we used the fact that $\mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v})$ for any vectors $\mathbf{u}, \mathbf{v}$ and scalar c.) This is a strange formula. We will explore its meaning in the next problem.

Problem 6. Universal Gravitation. Choose a coordinate system with a star at $(0,0,0)$ and let $\mathbf{r}(t)$ be the position of a planet at time t. Newton tells us that the planet feels a gravitational force $\mathbf{F}(t)$ pointed directly toward the sun. Specifically, we have

$$\mathbf{F}(t) = -\frac{GMm}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t),$$

where M is the mass of the star, m is the mass of the planet and G is a universal constant.

(a) Use Newton's Second Law, $\mathbf{F}(t) = m\mathbf{r}''(t)$, to show that $\mathbf{r}''(t) = (-GM/\|\mathbf{r}(t)\|^3)\mathbf{r}(t)$.

(b) The vector $\mathbf{h}(t) := \mathbf{r}(t) \times \mathbf{r}'(t)$ is called *angular momentum*. Use part (a) and Problem 5(b) to prove that $\mathbf{h}'(t) = \langle 0, 0, 0\rangle$ for all t. It follows that $\mathbf{h}(t)$ is a constant vector.

(c) Let's say that $\mathbf{h} = \langle h_1, h_2, h_3\rangle$ is the constant angular momentum vector of two-body system. Explain why the planet always stays within the plane $h_1x + h_2y + h_3z = 0$. This plane is called the *ecliptic*.

(a): Combining the two given equations gives

$$\begin{aligned} m\mathbf{r}''(t) &= \mathbf{F}(t) \\ m\mathbf{r}''(t) &= -\frac{GMm}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t) \\ \mathbf{r}''(t) &= -\frac{GM}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t). \end{aligned}$$

(b): Now we consider the angular momentum vector:

$$\mathbf{h}(t) = \mathbf{r}(t) \times \mathbf{r}'(t).$$

From part (a) we know that $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for the (complicated) scalar function $c(t) = -GM/\|\mathbf{r}(t)\|^3$, hence from Problem 5(b) we conclude that

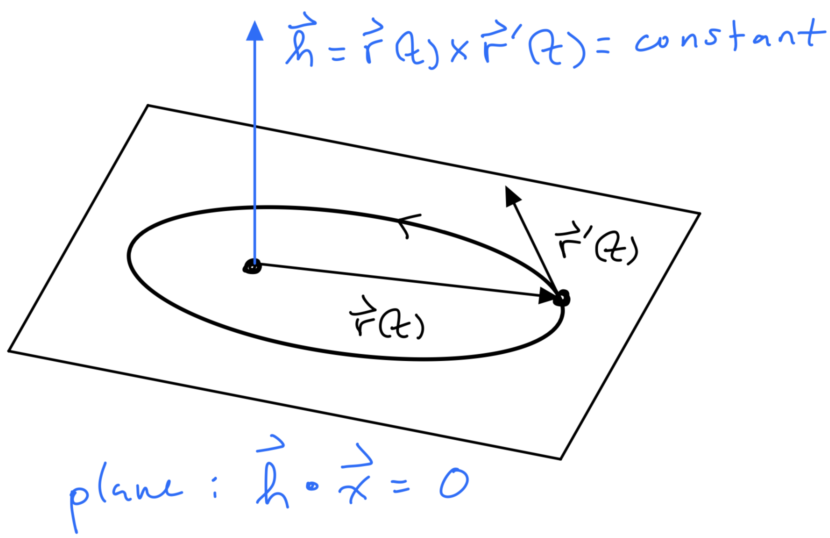
$$\mathbf{h}'(t) = [\mathbf{r}(t) \times \mathbf{r}'(t)]' = \langle 0, 0, 0\rangle.$$

In other words, the angular momentum vector $\mathbf{h}$ is constant. Let's write $\mathbf{h} = \langle h_1, h_2, h_3\rangle$.

(c): Now consider the plane $h_1x + h_2y + h_3z = 0$ that is perpendicular to the vector $\mathbf{h}$. I claim that the position of the planet $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ stays in this plane for all t. Indeed, we have defined $\mathbf{h}$ as the cross product $\mathbf{r}(t) \times \mathbf{r}'(t)$, which must be perpendicular to both $\mathbf{r}(t)$ and $\mathbf{r}'(t)$. In particular, we have

$$\begin{aligned} \mathbf{h} \bullet \mathbf{r}(t) &= 0 \\ \langle h_1, h_2, h_3\rangle \bullet \langle x(t), y(t), z(t)\rangle &= 0 \\ h_1x(t) + h_2y(t) + h_3z(t) &= 0. \end{aligned}$$

Here is a picture:



For the Curious Only! If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then the vector differential equation

$$\mathbf{r}''(t) = -\frac{GM}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t)$$

is equivalent to a system of three coupled nonlinear differential equations:

$$\begin{cases} x''(t) &= -GMx(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}, \\ y''(t) &= -GMy(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}, \\ z''(t) &= -GMz(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}. \end{cases}$$

One of Newton's great achievements was to show that these equations lead to the prediction of **elliptic planetary orbits**, which was earlier observed by Kepler without any explanation.

I will outline a modern proof of this using vector calculus. To simplify the formulas I will assume that $G = M = m = 1$.

- We showed in 6(b) that the angular velocity $\mathbf{h} = \mathbf{r}(t) \times \mathbf{r}'(t)$ is a constant vector.

- There is another conserved vector, called the *Runge-Lenz vector*:

  $$\mathbf{A}(t) = \mathbf{r}'(t) \times \mathbf{h} - \mathbf{r}(t)/\|\mathbf{r}(t)\|.$$

  One can check using identities for dot product and cross product that $\mathbf{A}'(t) = \langle 0, 0, 0\rangle$, hence $\mathbf{A}(t) = \mathbf{A}$ is constant. This is related to conservation of energy.

- Since $\mathbf{r}'(t) \times \mathbf{h}$ and $\mathbf{r}(t)/\|\mathbf{r}(t)\|$ are both perpendicular to $\mathbf{h}$, we see that $\mathbf{A}$ is perpendicular to $\mathbf{h}$. Thus we can choose a coordinate system so that $\mathbf{h} = \langle 0, 0, h\rangle$ and $\mathbf{A} = \langle e, 0, 0\rangle$ for some constants $e, h > 0$. The number e is some measure of energy.

- Since $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is perpendicular to $\mathbf{h} = \langle 0, 0, h\rangle$ we must have $z(t) = 0$ for all t. That is, the planet stays in the x, y-plane, which you showed in 6(c).

- Our goal is to find formulas for $x(t)$ and $y(t)$. This is much easier if we switch to polar coordinates $r(t)$ and $\theta(t)$ where $x(t) = r(t)\cos[\theta(t)]$ and $y(t) = r(t)\sin[\theta(t)]$. Note in particular that that $r(t) = \sqrt{x(t)^2 + y(t)^2} = \|\mathbf{r}(t)\|$. To save space we will write r and θ instead of $r(t)$ and $\theta(t)$.

- By computing the expression $\mathbf{r}(t) \bullet (\mathbf{r}'(t) \times \mathbf{h})$ in two different ways (using various identities for dot product and cross product) one can show that

  $$r(1 + e\cos\theta) = \mathbf{r}(t) \bullet (\mathbf{r}'(t) \times \mathbf{h}) = h^2,$$

  and hence

  $$r = h^2/(1 + e\cos\theta).$$

  This is the equation of a "conic section" in polar coordinates.
- The constant e is called the "eccentricity" of the orbit. If $0 < e < 1$ then the orbit is an ellipse. In this case we have the amazing property that the planet eventually returns to its original position.[4] It is certainly not obvious that this should happen! If $e > 1$ then the planet has enough energy to escape the solar system and the orbit is a hyperbola.

[4]Except it doesn't, because Newton's gravity is only approximately correct.