Math 310 **Spring 2024**
Homework 1 **Drew Armstrong**

Problem 1. Lines and Circles. The parametrized curve in part (a) is a line. The parametrized curve in part (b) is a circle. In each case, compute the velocity vector $\mathbf{f}'(t) = \langle x'(t), y'(t)\rangle$ and speed $\|\mathbf{f}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$ at time t. Also eliminate t to find an equation for the curve in terms of x and y. [Hint: In part (b) look at $(x-a)^2 + (y-b)^2$.]

(a) $\mathbf{f}(t) = (x(t), y(t)) = (p + ut, q + vt)$ where p, q, u, v are constants.
(b) $\mathbf{f}(t) = (x(t), y(t)) = (a + r\cos(\omega t), b + r\sin(\omega t))$ where a, b, r, ω are constants.

[Oops: The solution uses the letters a and b instead of p and q.]

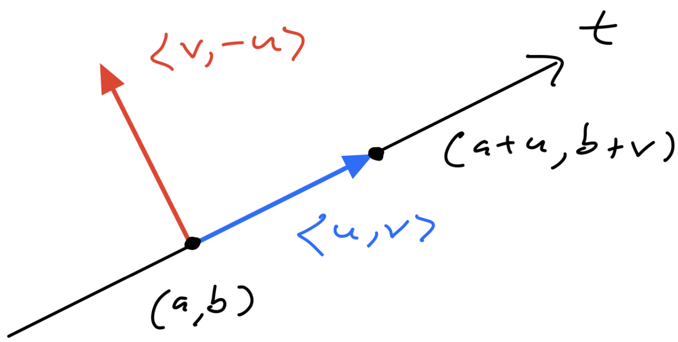
(a) **Line.** The velocity and speed are

$$(dx/dt, dy/dt) = (u, v) \quad \text{and} \quad \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{u^2 + v^2}.$$

Note that these are both constant, i.e., they do not depend on t. To eliminate t we will assume that $u \neq 0$ and $v \neq 0$, so that $x = a + ut$ implies $t = (x-a)/u$ and $y = b + vt$ implies $t = (y-b)/v$. Then equation these expressions for t gives

$$\begin{aligned} (x-a)/u &= (y-b)/v \\ v(x-a) &= u(y-b) \\ v(x-a) - u(y-b) &= 0. \end{aligned}$$

In class we will see that is the line that passes through the point (a, b) and is parallel to the vector $\langle u, v\rangle$. Equivalently, this line is **perpendicular** to the vector $\langle v, -u\rangle$:



(b) **Circle.** The velocity and speed are

$$(dx/dt, dy/dt) = (-r\omega\sin(\omega t), r\omega\cos(\omega t))$$

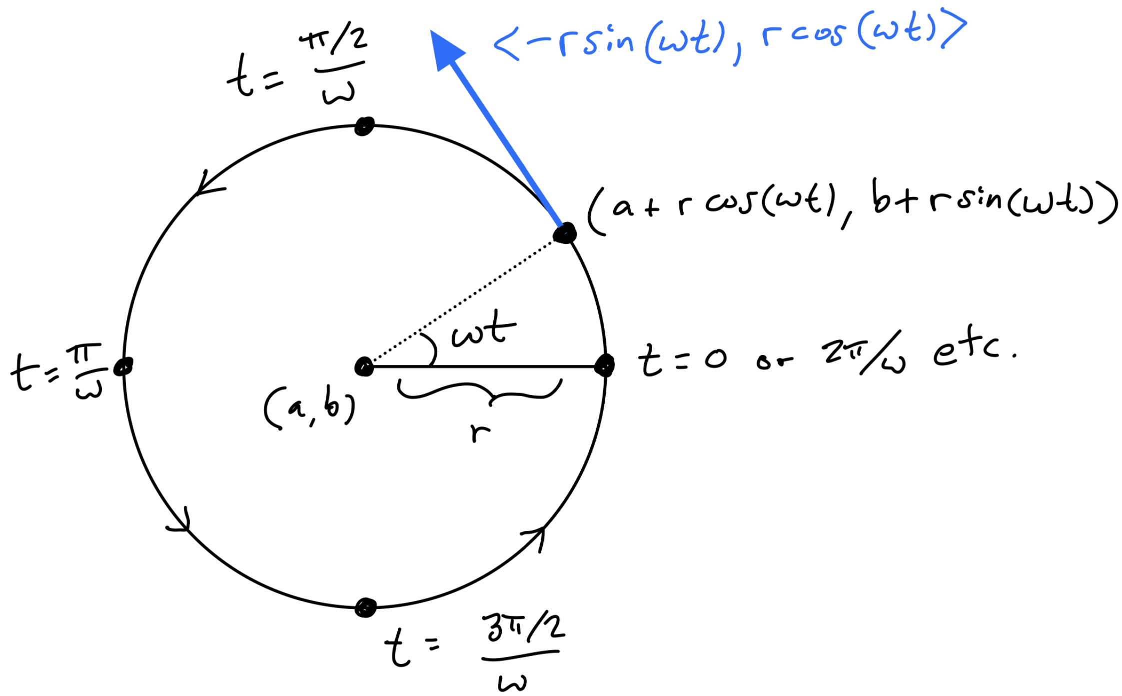
and

$$\begin{aligned} \sqrt{(dx/dt)^2 + (dy/dt)^2} &= \sqrt{[-r\omega\sin(\omega t)]^2 + [r\omega\cos(\omega t)]^2} \\ &= \sqrt{r^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} \\ &= \sqrt{r^2\omega^2} \\ &= r\omega. \end{aligned}$$

We assume that r and ω are positive, so $\sqrt{r^2\omega^2} = |r\omega| = r\omega$. The speed is constant, but the velocity vector is not constant.[1] We can eliminate t by using the trig identity $\sin^2(\omega t) + \cos^2(\omega t) = 1$ as follows:

$$(x-a)^2 + (y-b)^2 = [r\cos(\omega t)]^2 + [r\sin(\omega t)]^2$$
$$(x-a)^2 + (y-b)^2 = r^2[\cos^2(\omega t) + \sin^2(\omega t)]$$
$$(x-a)^2 + (y-b)^2 = r^2.$$

This is the equation of a circle with radius r, centered at (a, b). Here is a picture:



Problem 2. An Interesting Parametrized Curve. Consider the parametrized curve

$$\mathbf{f}(t) = (x(t), y(t)) = (t^2 - 1, t^3 - t).$$

(a) Compute the velocity vector $\mathbf{f}'(t) = \langle x'(t), y'(t)\rangle$ at time t.
(b) Find the slope of the tangent line at time t. [Hint: $dy/dx = (dy/dt)/(dx/dt)$.]
(c) Find all points on the curve where the tangent is horizontal or vertical.
(d) Sketch the curve. [Hint: Plot several points. Use a computer if you want.]
(e) Eliminate t to find an equation relating x and y. [Hint: This kind of problem is impossible in general, but in this case there is a very nice answer. Since $x = t^2 - 1$ we have $t = \pm\sqrt{x+1}$. Substitute this into y and simplify as much as possible.]

(a): The velocity vector is

$$\mathbf{f}'(t) = \langle x'(t), y'(t)\rangle = \langle 2t - 0, 3t^2 - 1\rangle = \langle 2t, 3t^2 - 1\rangle.$$

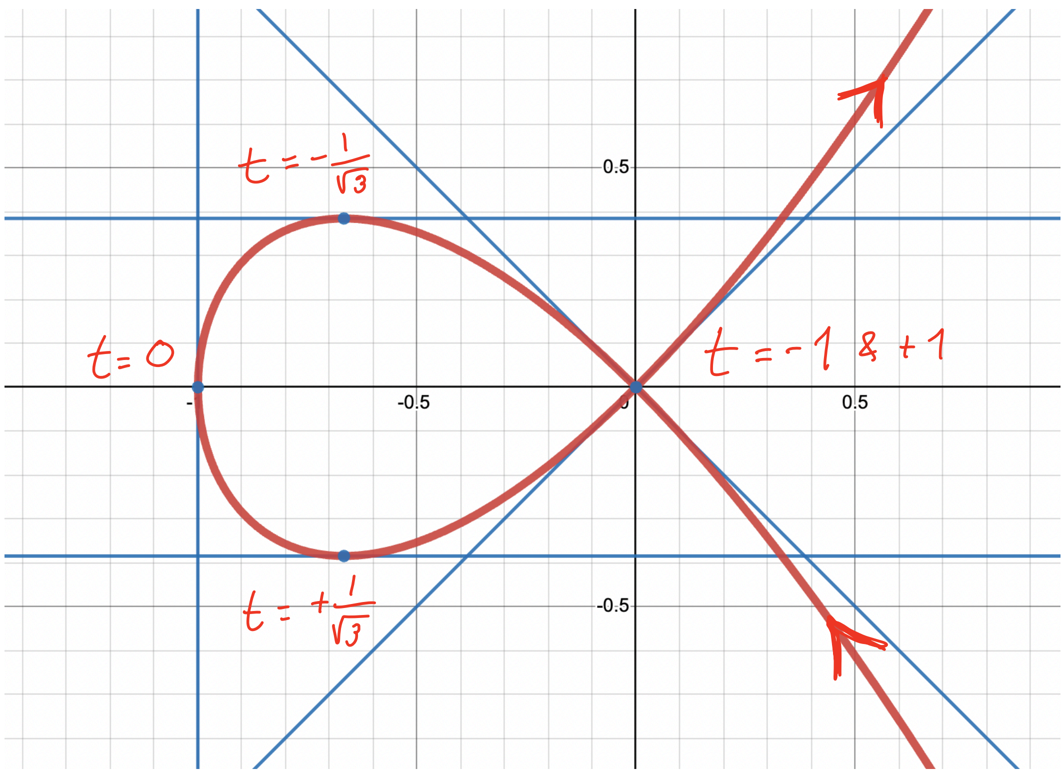
[1]There is different vector, called the *angular velocity*, that is constant. It points out of the page into the third dimension and it has length $r\omega$.

(b): The slope of the tangent line at time t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 1}{2t}.$$

(c): The tangent is horizontal when $dy/dx = 0$, which implies that $3t^2 - 1 = 0$, or $t = \pm 1/\sqrt{3}$. The tangent is vertical when dy/dx goes to $+\infty$ or $-\infty$. This happens when $t = 0$.

(d): Here is a picture:



Remark: This curve has the property that it crosses itself because $\mathbf{f}(-1) = (0,0) = \mathbf{f}(+1)$. At this point there are two tangent lines corresponding to the two times -1 and $+1$. The slopes of these two lines are $[3(-1)^2 - 1]/[2(-1)] = -1$ and $[3(+1)^2 - 1]/[2(+1)] = +1$.

(e): Since $x = t^2 - 1$ we have $t = \pm\sqrt{x+1}$. For simplicity let's take $t = \sqrt{x+1}$. Substituting this into $y = t^3 - t$ gives

$$\begin{aligned} y &= (\sqrt{x+1})^3 + \sqrt{x+1} \\ y &= \sqrt{x+1}\left((\sqrt{x+1})^2 - 1\right) \\ y &= \sqrt{x+1}(x+1-1) \\ y &= x\sqrt{x+1} \\ y^2 &= x^2(x+1) \\ y^2 &= x^3 + x^2. \end{aligned}$$

Note that taking $t = -\sqrt{x+1}$ would yield the same expression. One can check that this defines the same shape by plotting the equation $y^2 = x^3 + x^2$ in Desmos.

Problem 3. Arc Length. Consider the parametrized curve $\mathbf{f}(t) = (t^2, t^3)$. Find the arc length of this curve between times $t = 0$ and $t = 1$. [Hint: The arc length is the integral of the speed: $\int_0^1 \|\mathbf{f}'(t)\|\, dt$. Arc length is generally impossible to compute by hand but in this case there is a lucky accident that allows the integral to be computed via substitution.]

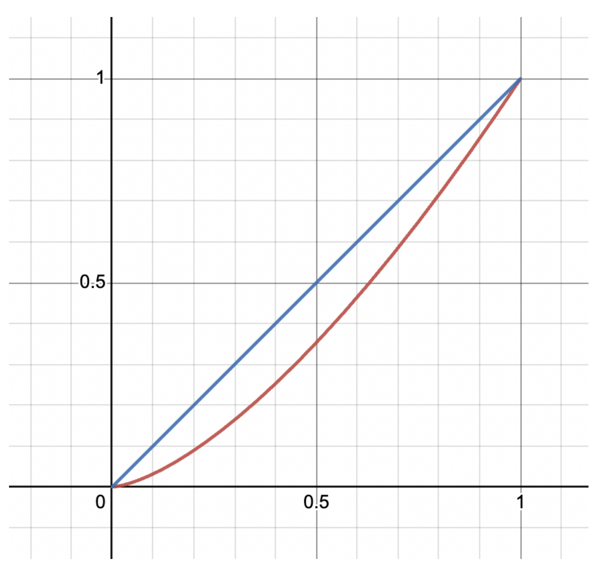
Solutions: The velocity is $\mathbf{f}'(t) = \langle 2t, 3t^2 \rangle$ and the speed is

$$\|\mathbf{f}'(t)\| = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4 + 9t^2)},$$

which we can write as $\|\mathbf{f}'(t)\| = t\sqrt{4 + 9t^2}$ when $t \geq 0$. It is a lucky coincidence that this function can be integrated by substitution:

$$\begin{aligned}
\text{arc length} &= \int_0^1 \|\mathbf{f}'(t)\|\, dt \\
&= \int_{t=0}^{t=1} t\sqrt{4+9t^2}\, dt \\
&= \frac{1}{18} \int_{u=4}^{u=13} \sqrt{u}\, du && (u = 4 + 9t^2, du = 18t\, dt) \\
&= \frac{1}{18} \cdot \frac{u^{3/2}}{3/2} \Bigg|_{u=4}^{u=13} \\
&= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) \\
&\approx 1.44.
\end{aligned}$$

Does this make sense? Here is a picture of the path (t^2, t^3), which travels from $(0, 0)$ to $(1, 1)$ as t goes from 0 to 1, and the straight line path between these points:



The blue straight line has length $\sqrt{2} \approx 1.41$, so the length of the red path must be slightly greater. So, yes, the answer 1.44 makes sense.

Problem 4. A Triangle in the Plane. Consider the following points in $\mathbb{R}^2$:

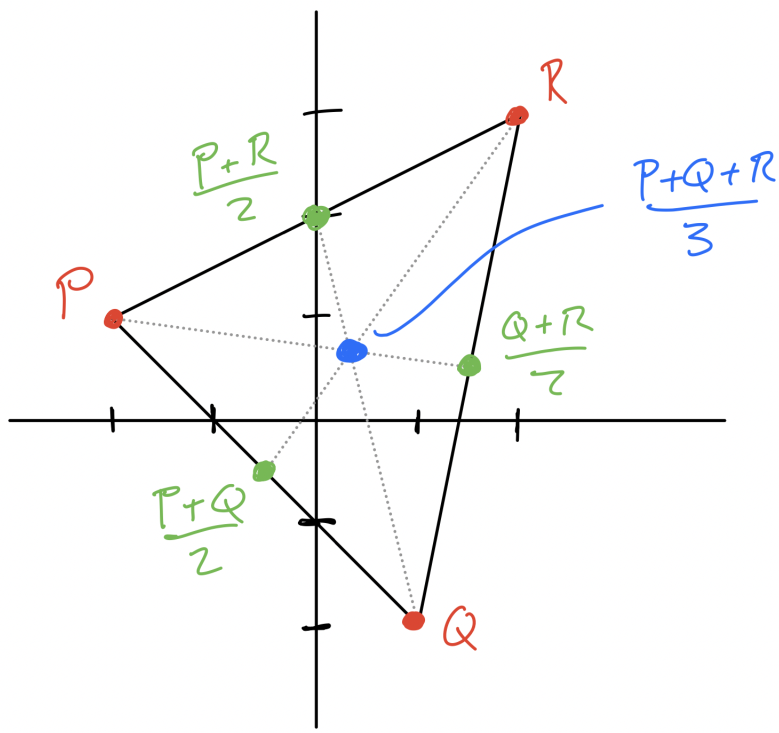
$$P = (-2,1), \quad Q = (1,-2), \quad R = (2,3).$$

(a) Draw the three points P, Q, R, the midpoints $(P+Q)/2$, $(P+R)/2$, $(Q+R)/2$ and the centroid $(P+Q+R)/3$.
(b) Find the coordinates of the three side vectors $\mathbf{u} = \vec{PQ}, \mathbf{v} = \vec{PR}, \mathbf{w} = \vec{QR}$. Check that $\mathbf{u} + \mathbf{w} = \mathbf{v}$. This is true because of the rule $\vec{PQ} + \vec{QR} = \vec{PR}$.
(c) Use the length formula to compute the three side lengths $\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|$.
(d) Use the dot product to compute the three angles of the triangle. After computing the angles, check that they sum to 180°. [Hint: Let α, β, γ be the angles at P, Q, R. The dot product theorem says that $\cos\alpha = \mathbf{u} \bullet \mathbf{v}/(\|\mathbf{u}\|\|\mathbf{v}\|)$. What about β and γ?]

(a): First we compute:

$$\begin{aligned}
(P+Q)/2 &= [(-2,1) + (1,-2)]/2 = (-1,-1)/2 = (-1/2,-1/2),\\
(P+R)/2 &= [(-2,1) + (2,3)]/2 = (0,4)/2 = (0,2),\\
(Q+R)/2 &= [(1,-2) + (2,3)]/2 = (3,1)/2 = (3/2,1/2),\\
(P+Q+R)/3 &= [(-2,1) + (1,-2) + (2,3)]/3 = (1,2)/3 = (1/3,2/3).
\end{aligned}$$

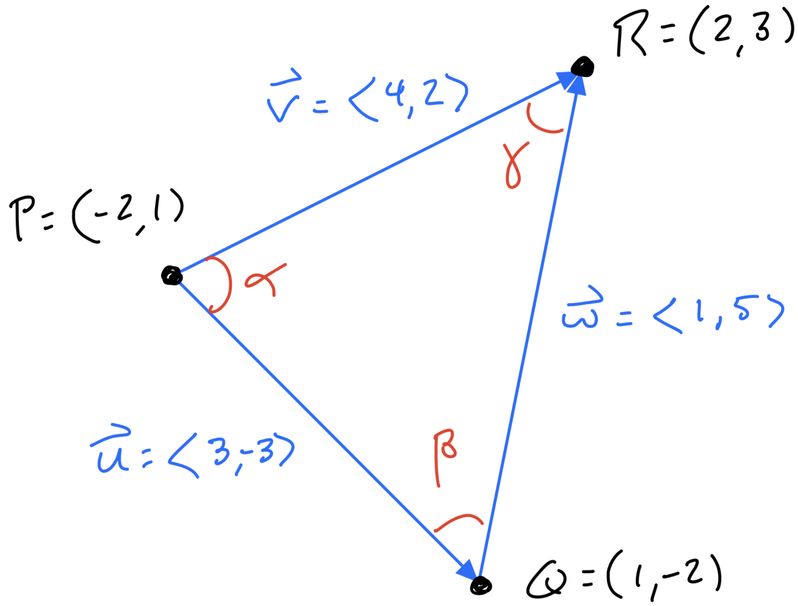
And then we draw:



(b): In coordinates, the vectors $\mathbf{u} = \vec{PQ}, \mathbf{v} = \vec{PR}, \mathbf{w} = \vec{QR}$ are

$$\begin{aligned}
\mathbf{u} &= P - Q = (1,-2) - (-2,1) = \langle 3,-3\rangle,\\
\mathbf{v} &= R - P = (2,3) - (-2,1) = \langle 4,2\rangle,\\
\mathbf{w} &= R - Q = (2,3) - (1,-2) = \langle 1,5\rangle.
\end{aligned}$$

Here is a picture:



From the alignment of the vectors, we see that $\mathbf{u} + \mathbf{w} = \mathbf{v}$, and the arithmetic checks out:

$$\mathbf{u} + \mathbf{w} = \langle 3, -3\rangle + \langle 1, 5\rangle = \langle 4, 2\rangle = \mathbf{v}.$$

(c): According to the Pythagorean Theorem, the side lengths are

$$\begin{aligned} \|\mathbf{u}\| &= \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{3^2 + (-3)^2} = \sqrt{18} \approx 4.24, \\ \|\mathbf{v}\| &= \sqrt{\mathbf{v} \bullet \mathbf{v}} = \sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47, \\ \|\mathbf{w}\| &= \sqrt{\mathbf{w} \bullet \mathbf{w}} = \sqrt{1^2 + 5^2} = \sqrt{26} \approx 5.10. \end{aligned}$$

(d): In order to compute the angles, we first compute the dot products:

$$\begin{aligned} \mathbf{u} \bullet \mathbf{v} &= (3)(4) + (-3)(2) = 6, \\ \mathbf{u} \bullet \mathbf{w} &= (3)(1) + (-3)(5) = -12, \\ \mathbf{v} \bullet \mathbf{w} &= (4)(1) + (2)(5) = 14. \end{aligned}$$

Since α is the angle between $\mathbf{u}$ and $\mathbf{v}$ (placed tail-to-tail), the dot product theorem says

$$\cos\alpha = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u} \bullet \mathbf{v}}{\sqrt{\mathbf{u} \bullet \mathbf{u}}\sqrt{\mathbf{v} \bullet \mathbf{v}}} = \frac{6}{\sqrt{18}\sqrt{20}}.$$

Similarly, since β is the angle between $-\mathbf{u}$ and $\mathbf{w}$ (placed tail-to-tail), we have

$$\cos\beta = \frac{(-\mathbf{u}) \bullet \mathbf{w}}{\|-\mathbf{u}\|\|\mathbf{w}\|} = \frac{-\mathbf{u} \bullet \mathbf{w}}{\sqrt{\mathbf{u} \bullet \mathbf{u}}\sqrt{\mathbf{w} \bullet \mathbf{w}}} = \frac{12}{\sqrt{18}\sqrt{26}},$$

and since γ is the angle between $-\mathbf{v}$ and $-\mathbf{w}$ (placed tail-to-tail) we have

$$\cos\gamma = \frac{(-\mathbf{v}) \bullet (-\mathbf{w})}{\|-\mathbf{v}\|\|-\mathbf{w}\|} = \frac{\mathbf{v} \bullet \mathbf{w}}{\sqrt{\mathbf{v} \bullet \mathbf{v}}\sqrt{\mathbf{w} \bullet \mathbf{w}}} = \frac{14}{\sqrt{20}\sqrt{26}}.$$

My computer says that $\alpha = 71.6°$, $\beta = 56.3°$ and $\gamma = 52.1°$, which, indeed, add up to $180°$.

Problem 5. Some Properties of Vector Arithmetic. Consider three vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \quad \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \quad \mathbf{w} = \langle w_1, w_2, w_3 \rangle.$$

(a) For any real number $a \in \mathbb{R}$ check that $(a\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (a\mathbf{v}) = a(\mathbf{u} \bullet \mathbf{v})$.
(b) Check the distributive property: $(\mathbf{u} + a\mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + a(\mathbf{v} \bullet \mathbf{w})$.
(c) Substitute $\mathbf{w} = \mathbf{u} + a\mathbf{v}$ in part (b) to show that

$$(\mathbf{u} + a\mathbf{v}) \bullet (\mathbf{u} + a\mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + a^2(\mathbf{v} \bullet \mathbf{v}) + 2a(\mathbf{u} \bullet \mathbf{v})$$

(d) Substitute $a = -1$ in part (c) to show that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

[Hint: Recall that $\|\mathbf{x}\|^2 = \mathbf{x} \bullet \mathbf{x}$ for any vector $\mathbf{x}$.]

(a): We explicit compute each of the three expressions:

$$\begin{aligned}
(a\mathbf{u}) \bullet \mathbf{v} &= \langle au_1, au_2, au_3 \rangle \bullet \langle v_1, v_2, v_3 \rangle = (au_1)v_1 + (au_2)v_2 + (au_3)v_3, \\
\mathbf{u} \bullet (a\mathbf{v}) &= \langle u_1, u_2, u_3 \rangle \bullet \langle av_1, av_2, av_3 \rangle = u_1(av_1) + u_2(av_2) + u_3(av_3), \\
a(\mathbf{u} \bullet \mathbf{v}) &= a(\langle u_1, u_2, u_3 \rangle \bullet \langle v_1, v_2, v_3 \rangle) = a(u_1v_1 + u_2v_2 + u_3v_3).
\end{aligned}$$

Note that each of these is equal to $au_1v_1 + au_2v_2 + au_3v_3$, so they are all the same.

(b): Expanding the left hand side gives

$$\begin{aligned}
(\mathbf{u} + a\mathbf{v}) \bullet \mathbf{w} &= (\langle u_1, u_2, u_3 \rangle + a\langle v_1, v_2, v_3 \rangle) \bullet \langle w_1, w_2, w_3 \rangle \\
&= \langle u_1 + av_1, u_2 + av_2, u_3 + av_3 \rangle \bullet \langle w_1, w_2, w_3 \rangle \\
&= (u_1 + av_1)w_1 + (u_2 + av_2)w_2 + (u_3 + av_3)w_3 \\
&= u_1w_1 + av_1w_1 + u_2w_2 + av_2w_2 + u_3w_3 + av_3w_3,
\end{aligned}$$

and expanding the right hand side gives

$$\begin{aligned}
\mathbf{u} \bullet \mathbf{w} + a(\mathbf{v} \bullet \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \bullet \langle w_1, w_2, w_3 \rangle + a(\langle v_1, v_2, v_3 \rangle \bullet \langle w_1, w_2, w_3 \rangle) \\
&= (u_1w_1 + u_2w_2 + u_3w_3) + a(v_1w_1 + v_2w_2 + v_3w_3) \\
&= u_1w_1 + u_2w_2 + u_3w_3 + av_1w_1 + av_2w_2 + av_3w_3,
\end{aligned}$$

which is the same thing.

(c): This time I won't write out all of the details. Instead, I will use a more abstract method by applying the result from part (b):[2]

$$\begin{aligned}
(\mathbf{u} + a\mathbf{v}) \bullet (\mathbf{u} + a\mathbf{v}) &= \mathbf{u} \bullet (\mathbf{u} + a\mathbf{v}) + a(\mathbf{v} \bullet (\mathbf{u} + a\mathbf{v})) && \text{from (b)} \\
&= \mathbf{u} \bullet \mathbf{u} + a(\mathbf{u} \bullet \mathbf{v}) + a(\mathbf{v} \bullet \mathbf{u} + a\mathbf{v} \bullet \mathbf{v}) && \text{from (b)} \\
&= \mathbf{u} \bullet \mathbf{u} + a(\mathbf{u} \bullet \mathbf{v}) + a(\mathbf{v} \bullet \mathbf{u}) + a^2(\mathbf{v} \bullet \mathbf{v}) \\
&= \mathbf{u} \bullet \mathbf{v} + 2a(\mathbf{u} \bullet \mathbf{v}) + a^2(\mathbf{v} \bullet \mathbf{v}).
\end{aligned}$$

(d): Substitute $a = -1$ into part (c) to get

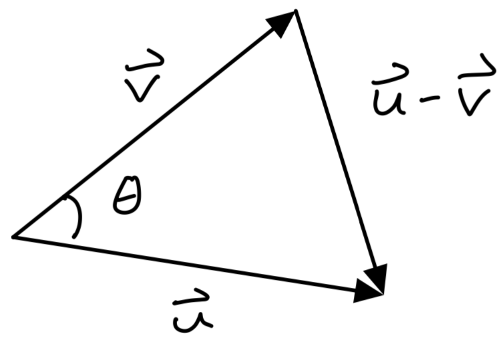
$$(\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = \mathbf{u} \bullet \mathbf{v} - 2(\mathbf{u} \bullet \mathbf{v}) + (-1)^2(\mathbf{v} \bullet \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} - 2(\mathbf{u} \bullet \mathbf{v}).$$

[2]We also need a few more basic rules: $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$, $a(b\mathbf{u}) = (ab)\mathbf{u}$ and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$. But these are easy to check so I won't bother.

Then use the formula $\mathbf{x} \bullet \mathbf{x} = \|\mathbf{x}\|^2$ to get

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

Remark: This algebraic formula is the key to the proof of the Dot Product Theorem. Consider any two vectors $\mathbf{u}$ and $\mathbf{v}$, placed tail-to-tail, with angle θ between them, and consider the vector $\mathbf{u} - \mathbf{v}$, which forms the third side of the triangle:



The geometric Law of Cosines says that the three side lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{u} - \mathbf{v}\|$ and the angle θ are related as follows:[3]

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\cos\theta.$$

On the other hand, we just proved that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

Since both of these formulas are true, we must have

$$\boxed{\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.}$$

[3]Note that the Law of Cosines becomes the Pythagorean Theorem when we set $\theta = 90^\circ$ becuase $\cos 90^\circ = 0$.