Math 310 **Exam 2**

Fall 2023 **Fri Dec 1**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

1. Integrating a Scalar Over a Rectangle.

(a) Integrate $f(x, y) = x + y$ over the rectangle with $-1 \leq x \leq 2$ and $1 \leq y \leq 3$.

$$\begin{aligned}
\iint f(x,y)\,dxdy &= \int_{y=1}^{3} \left(\int_{x=-1}^{2} (x+y)\,dx \right) dy \\
&= \int_{y=1}^{3} \left[\frac{x^2}{2} + xy \right]_{x=-1}^{2} dy \\
&= \int_{y=1}^{3} \left[\frac{4}{2} + 2y - \left(\frac{1}{2} - y \right) \right] dy \\
&= \int_{y=1}^{3} \left[\frac{3}{2} + 3y \right] dy \\
&= \left[\frac{3}{2}y + 3\frac{y^2}{2} \right]_{y=1}^{3} \\
&= \frac{9}{2} + \frac{27}{2} - \left(\frac{3}{2} + \frac{3}{2} \right) \\
&= 15.
\end{aligned}$$

Remark: We could view this as the volume of the region above the rectangle in the x, y-plane with $-1 \leq x \leq 2$ and $1 \leq y \leq 3$ and below the surface $z = x + y$.

(b) Compute the volume of the 3D region above the square in the x, y-plane with $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and below the surface $z = x^2 y$.

We can view $x^2 y\,dxdy$ as the volume of a skinny column above the point $(x, y, 0)$, where $x^2 y$ is the height of the column and $dxdy$ are the area of the base. Hence

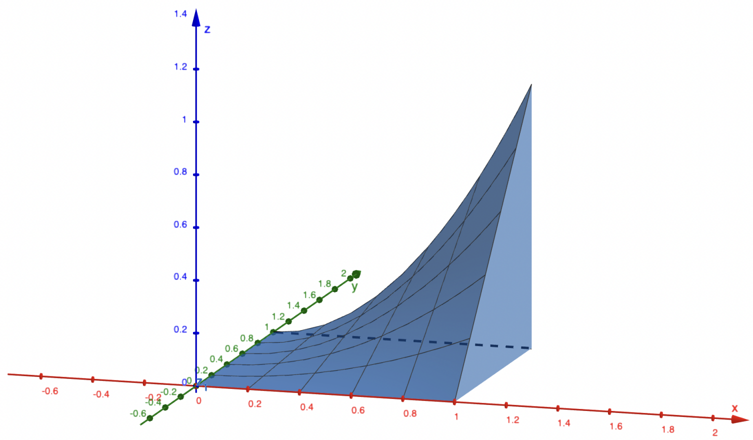
$$\begin{aligned}
\text{Volume} &= \iint (\text{skinny columns}) \\
&= \iint x^2 y\,dxdy \\
&= \int_0^1 x^2\,dx \cdot \int_0^1 y\,dy \\
&= \left(\frac{1^3}{3} - \frac{0^3}{3} \right) \cdot \left(\frac{1^2}{2} - \frac{0^2}{2} \right) \\
&= 1/6.
\end{aligned}$$

Alternatively, some students parametrized the 3D region by $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq x^2 y$ and then computed

$$\text{Volume} = \iiint 1\,dxdydz$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^{x^2 y} 1 \, dz \right) dx \right) dy$$
$$= \int_0^1 \left(\int_0^1 x^2 y \, dx \right) dy$$
$$= \text{same as before.}$$

Here is a picture of the 3D region:



2. Polar and Cylindrical Coordinates.

(a) Use polar coordinates to integrate $f(x, y) = x^2 + y^2$ over the unit disk $x^2 + y^2 \leq 1$.

Let $x = r\cos\theta$ and $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dxdy = r\,drd\theta$. The unit disk is parametrized by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, so that

$$\iint_{\text{disk}} (x^2 + y^2)\, dxdy = \iint_{\text{disk}} r^2 \cdot r\, drd\theta$$
$$= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 r^3\, dr$$
$$= 2\pi \cdot \left(\frac{1^4}{4} - \frac{0^4}{4} \right)$$
$$= \pi/2.$$

(b) Use cylindrical coordinates to integrate $f(x, y, z) = x^2 + y^2 + z^2$ over the cylinder satisfying $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

Let $x = r\cos\theta$ and $y = r\sin\theta$ so that $r^2 = x^2 + y^2$ and $dxdydz = r\,drd\theta dz$. The cylinder is parametrized by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$, so that

$$\iiint_{\text{cylinder}} (x^2 + y^2 + z^2)\, dxdydz$$
$$= \iiint_{\text{cylinder}} (r^2 + z^2) \cdot r\, drd\theta dz$$
$$= \iiint_{\text{cylinder}} (r^3 + rz^2)\, drd\theta dz$$

$$= \int_0^{2\pi} 1\, d\theta \cdot \int_{z=0}^1 \left(\int_{r=0}^1 (r^3 + rz^2)\, dr \right) dz$$
$$= 2\pi \cdot \int_0^1 \left[\frac{r^4}{4} + \frac{r^2}{2} z^2 \right]_{r=0}^1 dz$$
$$= 2\pi \cdot \int_0^1 \left(\frac{1}{4} + \frac{1}{2} z^2 \right) dz$$
$$= 2\pi \cdot \left[\frac{1}{4} z + \frac{1}{2} \cdot \frac{z^3}{3} \right]_{z=0}^1$$
$$= 2\pi \cdot \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \right)$$
$$= 5\pi/6.$$

3. Surface Area. Consider the following parametrized surface in 3D:

$$\mathbf{r}(u,v) = \langle u, v, u^2 + uv \rangle \quad \text{with } 0 \le u \le 1 \text{ and } 0 \le v \le 1.$$

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

We have $\mathbf{r}_u = \langle 1, 0, 2u + v \rangle$, $\mathbf{r}_v = \langle 0, 1, u \rangle$, and

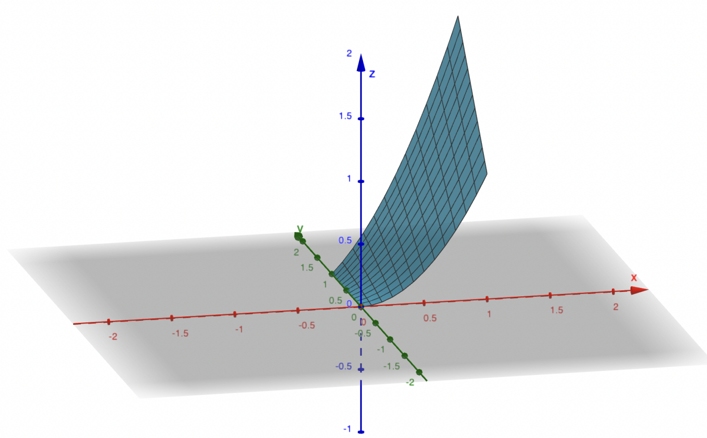
$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u - v, -u, 1 \rangle.$$

(b) Use your answer from part (a) to set up an integral to compute the **area of the surface** and simplify as much as possible. [This integral is too difficult to evaluate.]

Hence the surface area is

$$\iint 1 \|\mathbf{r}_u \times \mathbf{r}_v\|\, du dv = \int_0^1 \int_0^1 \sqrt{(-2u - v)^2 + (-u)^2 + 1^2}\, du dv$$
$$= \int_0^1 \int_0^1 \sqrt{5u^2 + 4uv + v^2 + 1}\, du dv.$$

This cannot be evaluated by hand. My computer gives 1.91994. Here is a picture:



4. Green's Theorem. Consider the vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle -y^3/3, x^3/3 \rangle$.

(a) Integrate the scalar $\text{curl}(\mathbf{F}) = Q_x - P_y$ over the unit disk $x^2 + y^2 \le 1$.

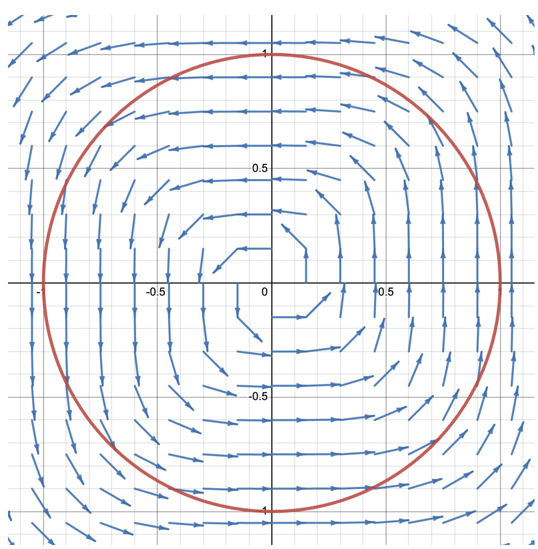
Since $Q_x - P_y = 3x^2/3 - (-3y^2/3) = x^2 + y^2$, this problem is the same as Problem 2(a). The answer is $\pi/2$.

(b) Set up the line integral of the vector field $\mathbf{F}$ around the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$, and simplify as much as possible. [This integral is difficult to evaluate directly, but Green's Theorem tells us that (a) and (b) have the same answer.]

The definition of the line integral gives

$$\begin{aligned}
\int \mathbf{F} \bullet \mathbf{T}\, ds &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t \rangle\, dt \\
&= \int_0^{2\pi} \langle -\sin^3 t/3, \cos^3 t/3 \rangle \bullet \langle -\sin t, \cos t \rangle\, dt \\
&= \int_0^{2\pi} \left(\frac{\sin^4 t}{3} + \frac{\cos^4 t}{3} \right) dt.
\end{aligned}$$

I guess this could be evaluated by hand, but it would take a while. The answer is $\pi/2$. Here is a picture of the disk and the vector field $\mathbf{F} = \langle -y^3/3, x^3/3 \rangle$:



5. Conservative Vector Fields. Consider the vector field $\mathbf{F}(x, y) = \langle P, Q \rangle = \langle xy^2, x^2y \rangle$. Note that this field is conservative because $Q_x = 2xy = P_y$.

(a) Find a scalar function $f(x, y)$ such that $\nabla f(x, y) = \mathbf{F}(x, y)$. [Hint: Compute the line integral of $\mathbf{F}$ along any parametrized path ending at the point (x, y).]

Let $f(x,y)$ be the line integral of $\mathbf{F}$ along the path $\langle xt, yt\rangle$ for $0 \le t \le 1$:

$$\begin{aligned}
f(x,y) &= \int_0^1 \mathbf{F}(xt,yt) \bullet \langle xt, yt\rangle' \, dt \\
&= \int_0^1 \langle (xt)(yt)^2, (xt)^2(yt)\rangle \bullet \langle xt, yt\rangle' \, dt \\
&= \int_0^1 \langle xy^2t^3, x^2yt^3\rangle \bullet \langle x, y\rangle \, dt \\
&= \int_0^1 \left(x^2y^2t^3 + x^2y^2t^3\right) dt \\
&= 2x^2y^2 \cdot \int_0^1 t^3 \, dt \\
&= 2x^2y^2 \cdot (1/4) \\
&= x^2y^2/2.
\end{aligned}$$

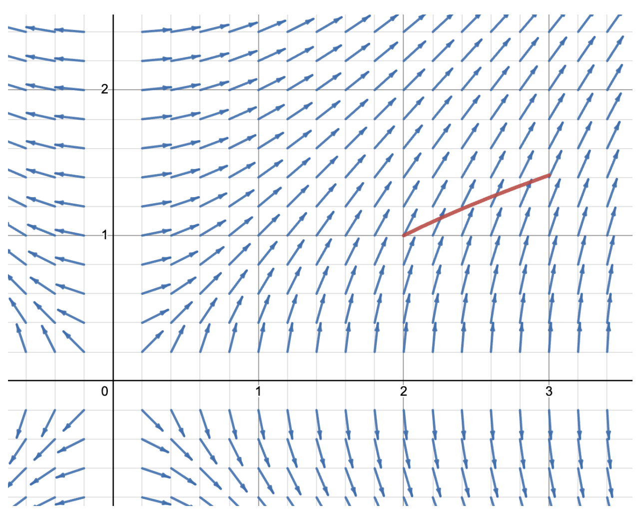
Then we check that $\nabla f = \nabla(x^2y^2/2) = \langle xy^2, x^2y\rangle = \mathbf{F}$ as desired.

(b) Use your answer from part (a) and the Fundamental Theorem of Line Integrals to compute the line integral of $\mathbf{F}$ along the path $\mathbf{r}(t) = \langle 1+t, \sqrt{t}\rangle$ for $1 \le t \le 2$.

Consider the path $\mathbf{r}(t) = \langle 1+t, \sqrt{t}\rangle$. (This is different from the path in part (a).) Since $\mathbf{F} = \nabla f$, the Fundamental Theorem of Line Integrals tells us that

$$\begin{aligned}
\int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt &= \int_1^2 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt \\
&= f(\mathbf{r}(2)) - f(\mathbf{r}(1)) \\
&= f(3, \sqrt{2}) - f(2, 1) \\
&= (3)^2(\sqrt{2})^2/2 - (2)^2(1)^2/2 \\
&= 9 - 2 \\
&= 7.
\end{aligned}$$

(We could also compute this without using the Fundamental Theorem, but it would take longer.) Here is a picture of the vector field $\mathbf{F} = \langle xy^2, x^2y\rangle$ and the path $\mathbf{r}$:



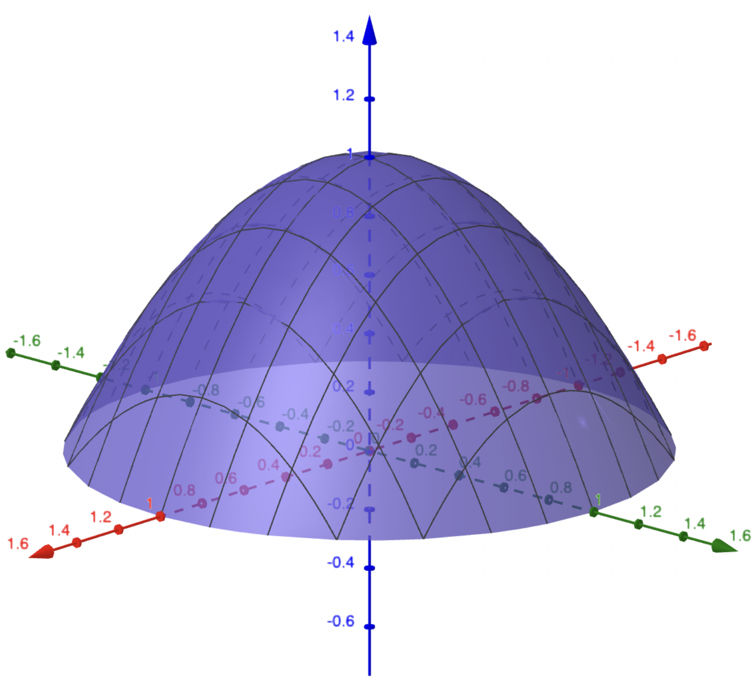
Math 211 **Summer 2022**
Quiz 4 **Drew Armstrong**

Problem 1. Double Integrals. Use polar coordinates to integrate the scalar function $f(x,y) = 1 - x^2 - y^2$ over the unit disk $x^2 + y^2 \leq 1$.

Polar coordinates are defined by $x = r\cos\theta$ and $y = r\sin\theta$, with $dxdy = rdrd\theta$. This is a good choice because the function f and the domain of integration both have rotational symmetry. To be specific, we have $f = 1 - x^2 - y^2 = 1 - r^2$, and the domain is parametrized by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Hence the integral is

$$\begin{aligned}
\iint_{\text{disk}} f &= \iint_{\text{disk}} (1-r^2) r dr d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 (r - r^3)\, dr \\
&= 2\pi \cdot \left[\frac{1}{2}r - \frac{1}{4}r^4\right]_0^1 \\
&= 2\pi\left[\frac{1}{2} - \frac{1}{4}\right] \\
&= \frac{\pi}{2}.
\end{aligned}$$

Remark: If we want, we could interpret this as the volume between the z-axis and the parabolic dome $z = 1 - x^2 - y^2$:



Problem 2. Triple Integrals.

(a) Find a parametrization for the tetrahedron in $\mathbb{R}^3$ with vertices

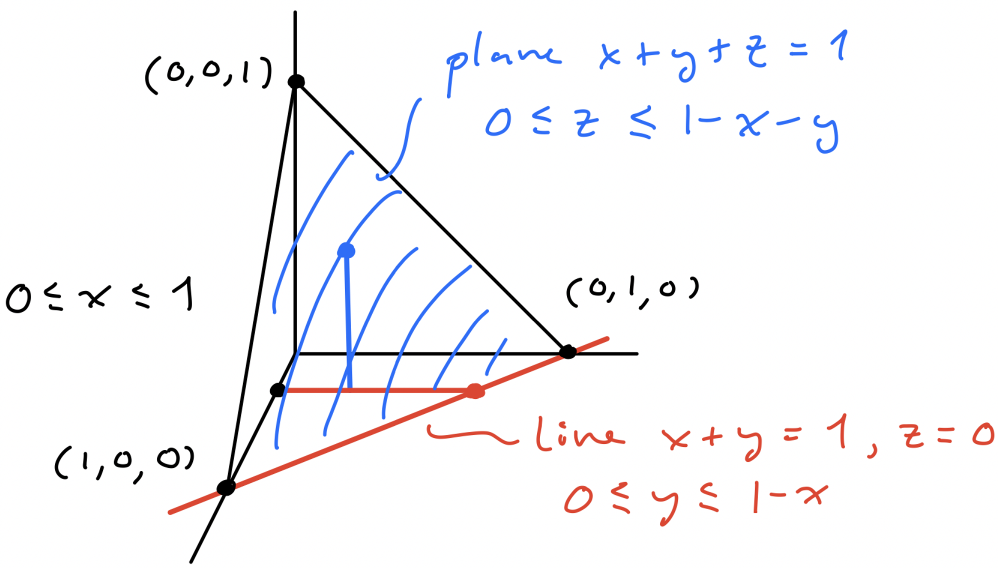
$$(0,0,0), \quad (1,0,0), \quad (0,1,0), \quad \text{and} \quad (0,0,1).$$

(b) Use your parametrization to compute the volume of the tetrahedron.

(a): If we choose x, then y, then z, we obtain the following parametrization:

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1, \\ 0 & \leq & y & \leq & 1-x, \\ 0 & \leq & z & \leq & 1-x-y. \end{array}$$

Here is a picture:



(b): The volume is

$$\begin{aligned} \iiint_{\text{tetrahedron}} 1\, dxdydz &= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} dz \right) dy \right) dx \\ &= \int_0^1 \left(\int_0^{1-x} (1-x-y)\, dy \right) dx \\ &= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} dx \\ &= \int_0^1 \left((1-x)(1-x) - \frac{1}{2}(1-x)^2 \right) dx \\ &= \int_0^1 \frac{1}{2}(1-x)^2\, dx \\ &= \left[-\frac{1}{6}(1-x)^3 \right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

Math 211 **Summer 2022**
Quiz 5 **Drew Armstrong**

Problem 1. Surface Area. Compute the area of the parametrized surface

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle = \langle u, u, v^2\rangle$$

for $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

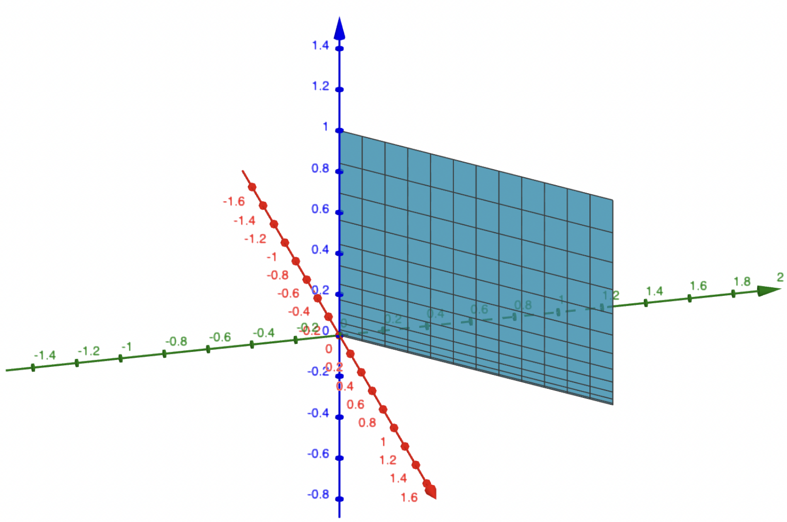
First we compute the stretch factor:

$$\begin{aligned}
\mathbf{r}_u &= \langle 1, 1, 0\rangle, \\
\mathbf{r}_v &= \langle 0, 0, 2v\rangle, \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle 2v, -2v, 0\rangle, \\
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{4v^2 + 4v^2} \\
&= \sqrt{8v^2} \\
&= 2\sqrt{2}v \text{ because } v \geq 0.
\end{aligned}$$

Then we compute the area:

$$\begin{aligned}
\iint 1\, dA &= \iint \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv \\
&= \iint 2\sqrt{2}v\, dudv \\
&= 2\sqrt{2} \cdot \int_0^1 du \cdot \int_0^1 v\, dv \\
&= 2\sqrt{2}(1)(1/2) \\
&= \sqrt{2}.
\end{aligned}$$

Remark: It's really hard to find a surface whose area is computable by hand. This surface is secretly just a rectangle with base $\sqrt{2}$ and height 1:



Problem 2. Line Integrals. Integrate the vector field[1] $\mathbf{F}(x, y) = \langle -y, x\rangle$ around the unit circle $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \leq t \leq 2\pi$.

From the definition we have

$$\begin{aligned}
\int_{\text{circle}} \mathbf{F} \bullet \mathbf{T} &= \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t\rangle\, dt \\
&= \int \langle -\sin t, \cos t\rangle \bullet \langle -\sin t, \cos t\rangle\, dt \\
&= \int \left[\sin^2 t + \cos^2 t\right]\, dt \\
&= \int_0^{2\pi} 1\, dt \\
&= 2\pi.
\end{aligned}$$

Remark: Since this integral is not zero, we conclude that the vector field $\mathbf{F}$ is not conservative.

Problem 3. Conservative Vector Fields. Find a scalar field $f(x, y)$ such that

$$\nabla f(x, y) = \langle 2x + 2y, 2x + 2y\rangle.$$

We will use the Fundamental Theorem of Line Integrals (or whatever you want to call it). Consider the path $\mathbf{r}(t) = \langle xt, yt\rangle$ for t from 0 to 2π. Then we have

$$\begin{aligned}
f(x, y) - f(0, 0) &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\
&= \int_0^1 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^1 \langle 2xt + 2yt, 2xt + 2yt\rangle \bullet \langle x, y\rangle\, dt \\
&= \int_0^1 \left[(2xt + 2yt)x + (2xt + 2yt)y\right]\, dt \\
&= \int_0^1 (2x^2 + 4xy + 2y^2)t\, dt \\
&= (2x^2 + 4xy + 2y^2) \cdot \int_0^1 t\, dt \\
&= (2x^2 + 4xy + 2y^2) \cdot (1/2) \\
&= x^2 + 2xy + y^2.
\end{aligned}$$

We conclude that $f(x, y) = x^2 + 2xy + y^2$, plus an arbitrary constant. Check:

$$\begin{aligned}
(x^2 + 2xy + y^2)_x &= 2x + 2y + 0, \\
(x^2 + 2xy + y^2)_y &= 0 + 2x + 2y.
\end{aligned}$$

[1] That is, integrate the component of $\mathbf{F}$ that is tangent to the curve. You know, the usual thing.

Quiz 4 Solutions:

1. Use cartesian coordinates to integrate the function $f(x,y) = x\sin(y)$ over the rectangle $0 \le x \le 3$, $0 \le y \le \pi$.

$$\iint x\sin(y)\,dx\,dy$$

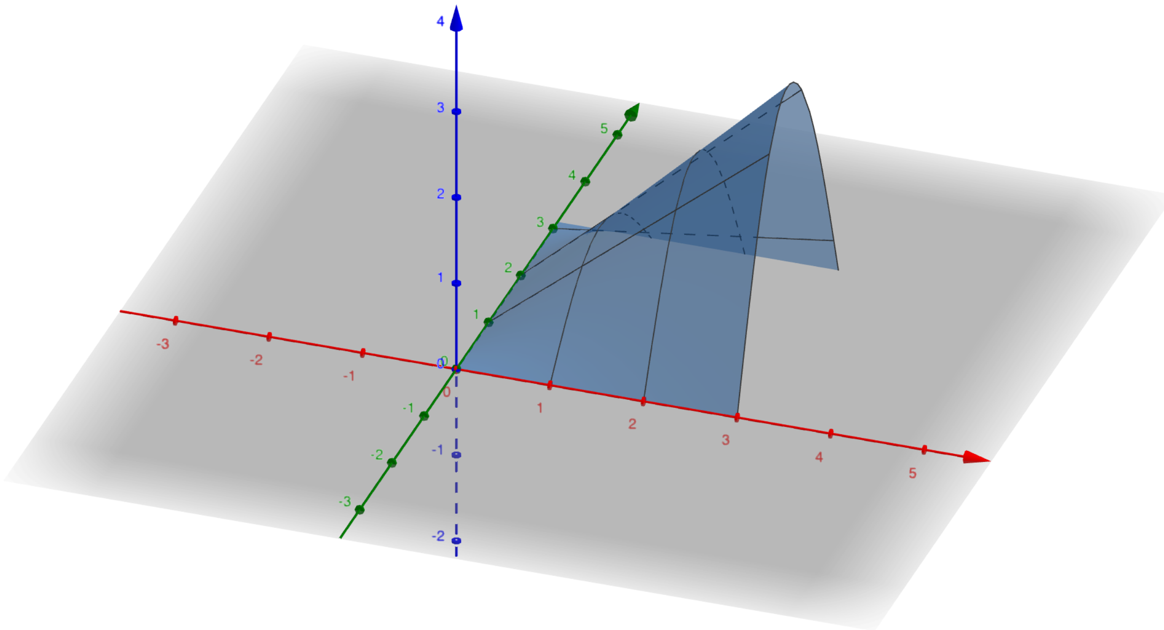
$$= \int_0^3 x\,dx \int_0^\pi \sin(y)\,dy$$

$$= \left(\tfrac{1}{2}x^2\right)_0^3 \cdot \left(-\cos(y)\right)_0^\pi$$

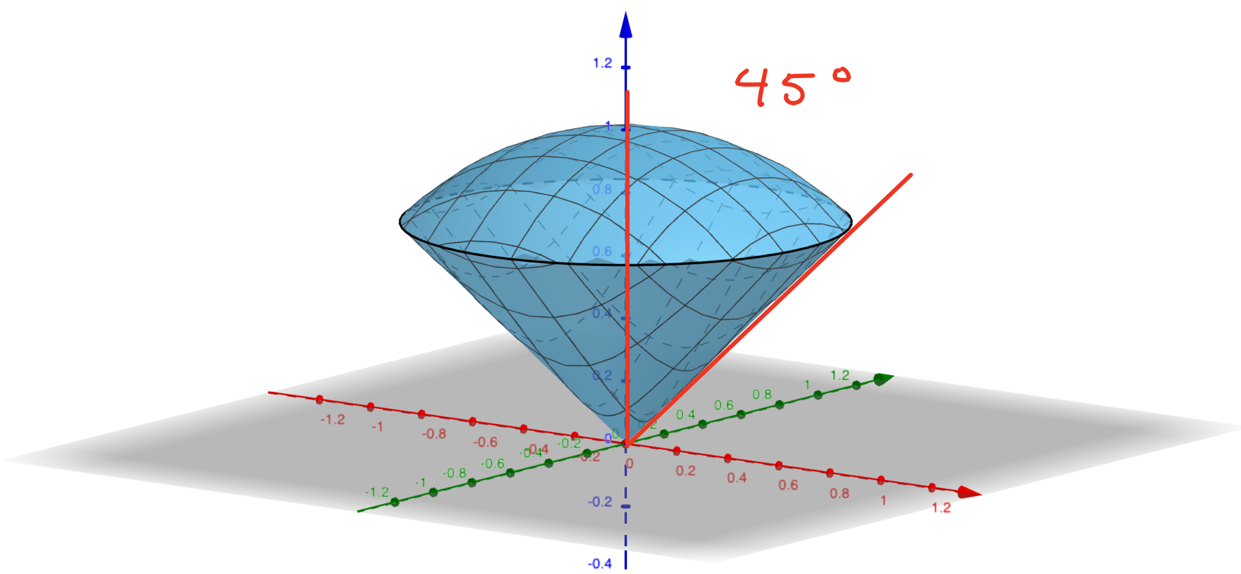
$$= \tfrac{1}{2}\cdot 9\left(-\cos(\pi) + \cos(0)\right) \quad [\cos(\pi) = -1,\ \cos(0) = +1]$$

$$= 9$$

If f = "height" then we can think of this integral as the volume of the region above the rectangle and below the surface $z = x\sin(y)$:



2. Use spherical coordinates to find the volume of the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $z = \sqrt{1-x^2-y^2}$:



we have $0 \le \theta \le 2\pi$ & $0 \le \varphi \le \pi/4$.

The sphere has radius 1 and is centered at the origin, so $0 \le \rho \le 1$.

Therefore the volume is

$$\text{vol} = \iiint dV$$

$$= \iiint \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^2 d\rho \int_0^{\pi/4} \sin\varphi \, d\varphi$$

$$= 2\pi \cdot \frac{1}{3} \cdot \left(-\cos\left(\frac{\pi}{4}\right) + \cos(0)\right)$$

$$= \frac{2\pi}{3}\left(1 - \sqrt{2}/2\right)$$

[About 30% of the volume of the full hemisphere, which is $\frac{2}{3}\pi$.]

Quiz 5 Solutions:

1. Consider $f(x,y,z) = xy + z$. Compute the integral of the gradient vector field $\vec{F} = \nabla f$ along the path $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $t=0$ to $t=1$.

Easy Way (Fundamental Theorem of line integrals):

$$\int_0^1 \nabla f(\vec{r}(t)) \circ r'(t)\, dt$$

$$= f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,1,1) - f(0,0,0)$$

$$= (1\cdot 1 + 1) - (0\cdot 0 + 0)$$

$$= 2.$$

Hard Way: First compute

$$\nabla f(x,y,z) = \langle y, x, 1 \rangle,$$

$$\nabla f(\vec{r}(t)) = \langle t^3, t, 1 \rangle, \quad \text{and}$$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle .$$

Then we have

$$\int_0^1 \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle t^2, t, 1 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt$$

$$= \int_0^1 (t^2 + 2t^2 + 3t^2)\, dt$$

$$= \int_0^1 6t^2\, dt$$

$$= 6\left(\tfrac{1}{3}t^3\right)\Big|_0^1 = 2 \quad \checkmark$$

2. Compute the circlutation of

$$\vec{F}(x,y) = \left\langle -\frac{y}{2} + e^x, \frac{x}{2} - \ln(y) \right\rangle$$

around the circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$ from $t = 0$ to $t = 2\pi$.

Easy Way (Green's Theorem):

First compute the curl

$$Q_x - P_y = \left(\frac{1}{2} + 0\right) - \left(-\frac{1}{2} + 0\right) = 1$$

If D is the interior of the unit circle C then Green's Theorem says

$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_D$ ~~curl($\vec{F}$)~~ 1 dA

$$= \iint_D dA$$

= area of unit circle

$= \pi$.

Hard Way: Since

$$\vec{F}(\vec{r}(t)) = \left\langle -\frac{\sin t}{2} + e^{\cos t}, \frac{\cos t}{2} - \ln(\sin t) \right\rangle$$

and

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle,$$

we have

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \left[\frac{\sin^2 t}{2} - \sin t\, e^{\cos t} + \frac{\cos^2 t}{2} - \cos t \ln(\sin t) \right] dt$$

$\vdots$ hard to do by hand

$= \pi$ (via computer) ✓