Math 310 **Exam 1**
Spring 2024 **Thurs Feb 29**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

Problem 1. Consider the line L in $\mathbb{R}^3$ that passes through the points

$$(1,-3,2) \text{ and } (4,2,3).$$

(a) Write down a parametrization for the line L.

We need one point on the line and a vector in the line. We will choose the point $(x_0, y_0, z_0) = (1,-3,2)$ and the vector $\langle u,v,w\rangle = (4,2,3)-(1,-3,2) = \langle 3,5,1\rangle$, to obtain the parametrization

$$\begin{aligned}\mathbf{r}(t) &= (1,-3,2) + t\langle 3,5,1\rangle \\ &= (1+3t, -3+5t, 2+t).\end{aligned}$$

(b) Write down the equations of two planes whose intersection is L.

From part (a) we have

$$\begin{cases} x &= 1+3t, \\ y &= -3+5t, \\ z &= 2+t, \end{cases} \quad\rightsquigarrow\quad \begin{cases} t &= (x-1)/3, \\ t &= (y+3)/5, \\ t &= z-2. \end{cases}$$

Eliminating t gives three equations:

$$(x-1)/3 = (y+3)/5, \quad (x-1)/3 = z-2, \quad (y+3)/5 = z-2.$$

Each of these is the equation of a plane that contains the line L. Pick two.

Remark: Both parts of Problem 1 have infinitely many correct solutions.

Problem 2. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be the trajectory of a particle in the x,y-plane. Suppose that the acceleration at time t is $\mathbf{r}''(t) = \langle 6t, 2\rangle$.

(a) Find the position $\mathbf{r}(t)$ at time t if the initial position is $\mathbf{r}(0) = (0,0)$ and the initial velocity is $\mathbf{r}'(0) = \langle 1,1\rangle$.

Integrating once gives

$$\begin{aligned}\mathbf{r}'(t) &= \int \mathbf{r}''(t)\,dt \\ &= \left\langle \int 6t\,dt, \int 2\,dt \right\rangle \\ &= \left\langle 3t^2 + c_1, 2t + c_2 \right\rangle.\end{aligned}$$

Substituting $t = 0$ gives $\mathbf{r}'(0) = \langle c_1, c_2\rangle$, so the given initial velocity implies that $c_1 = c_2 = 1$, and hence

$$\mathbf{r}'(t) = \left\langle 3t^2+1, 2t+1 \right\rangle.$$

Integrating again gives

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt \\ &= \left\langle \int (3t^2+1)\,dt, \int (2t+1)\,dt \right\rangle \\ &= \left\langle t^2+t+c_3, t^2+t+c_4 \right\rangle.\end{aligned}$$

Substituting $t = 0$ gives $\mathbf{r}(0) = \langle c_3, c_4 \rangle$, so the given initial position implies that $c_3 = c_4 = 0$. We conclude that

$$\mathbf{r}(t) = \left\langle t^3+t, t^2+t \right\rangle.$$

(b) Compute the slope of the tangent line at the point $\mathbf{r}(t)$. Use this to find all points on the curve where the tangent line is horizontal.

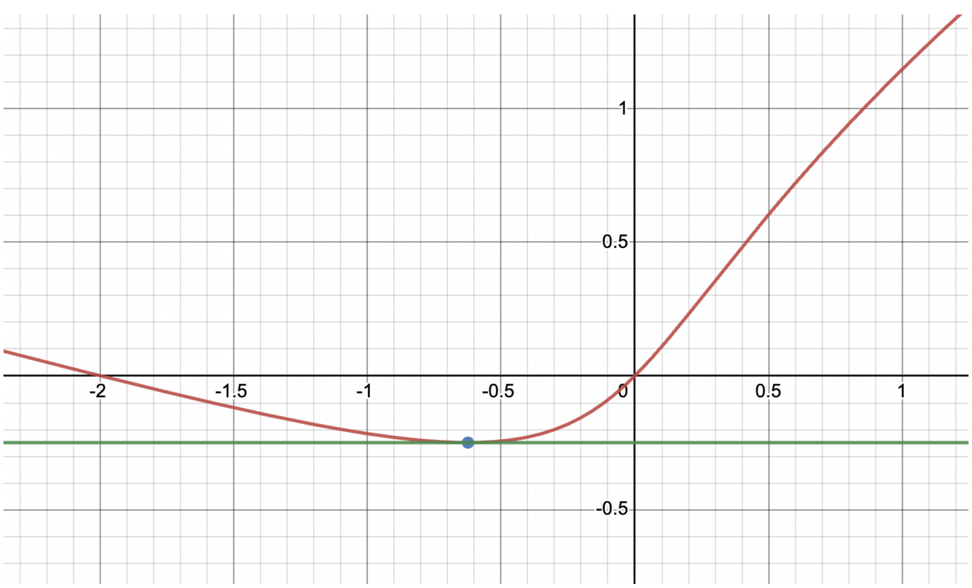
The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{3t^2+1}.$$

The tangent line is horizontal when $dy/dx = 0$, which implies that $2t+1 = 0$ and hence $t = -1/2$. The corresponding point is

$$\mathbf{r}(-1/2) = \left\langle \left(\frac{-1}{2}\right)^3 - \frac{1}{2}, \left(\frac{-1}{2}\right)^2 - \frac{1}{2} \right\rangle = \left\langle -\frac{5}{8}, -\frac{1}{4} \right\rangle.$$

Here is a picture:



Problem 3. Consider a parallelogram that has side lengths a and b with an angle of θ between them. The area of the parallelogram is $A = ab \sin\theta$.

(a) Use the chain rule to write an approximate formula for the uncertainty ΔA in terms of the uncertainties Δa, Δb and $\Delta \theta$.

The chain rule says that

$$\Delta A \approx \frac{\partial A}{\partial a}\Delta a + \frac{\partial A}{\partial b}\Delta b + \frac{\partial A}{\partial \theta}\Delta\theta$$
$$= b\sin\theta\Delta a + a\sin\theta\Delta b + ab\cos\theta\Delta\theta.$$

(b) Suppose that you measure the quantities $a = 10$ cm, $b = 15$ cm and $\theta = \pi/6$ radians. If your measuring equipment has uncertainties $\Delta a = \Delta b = 0.1$ cm and $\Delta\theta = \pi/180$ radians, use your formula from part (a) to estimate the uncertainty ΔA. You can leave your answer in unsimplified form.

Substituting $a = 10$, $b = 15$, $\theta = \pi/6$, $\Delta a = \Delta b = 0.1$ and $\Delta\theta = \pi/180$ gives

$$\begin{aligned}\Delta A &\approx b\sin\theta\Delta a + a\sin\theta\Delta b + ab\cos\theta\Delta\theta \\ &= (15)\sin(\pi/6)(0.1) + (10)\sin(\pi/6)(0.1) + (10)(15)\cos(\pi/6)(\pi/180) \\ &= (15)(1/2)(0.1) + (10)(1/2)(0.1) + (10)(15)(\sqrt{3}/2)(\pi/180) \\ &\approx 3.5 \text{ cm}^2.\end{aligned}$$

Remark: The computed area is $A = (10)(15)\sin(\pi/6) = 75$ cm^2, so the relative error is $\Delta A/A = 3.5/75 \approx 4.7\%$.

Problem 4. Consider the scalar field $f(x, y, z) = x^2 + y^4 + z^2x$.

(a) Compute the gradient vector field ∇f.

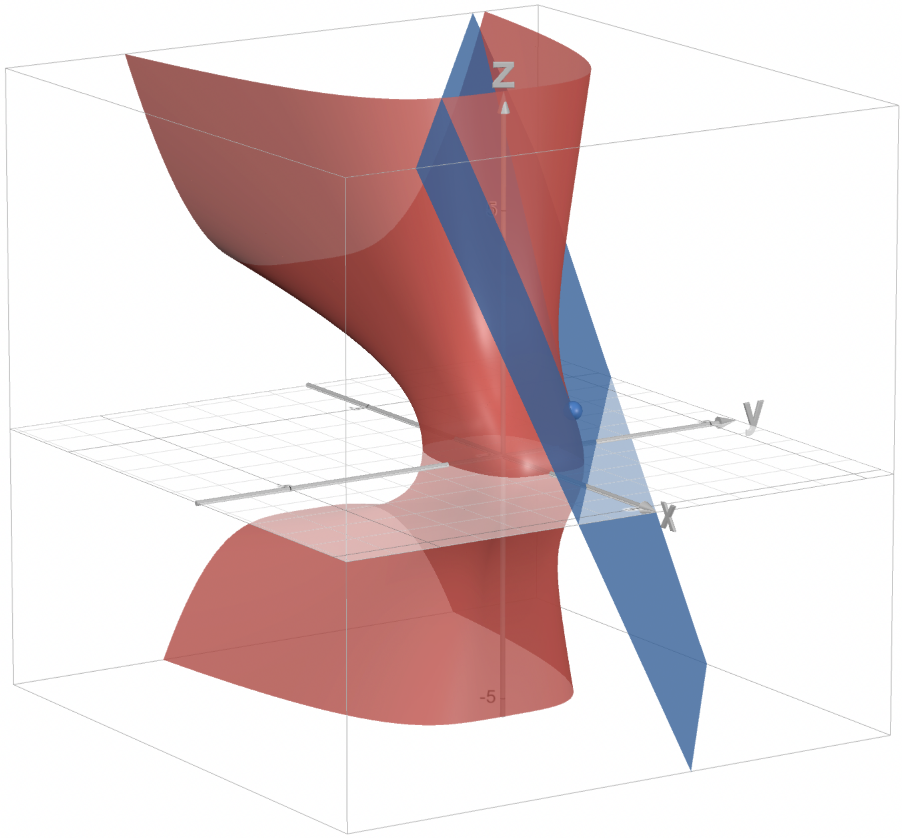
$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x, f_y, f_z\rangle \\ &= \left\langle 2x + z^2, 4y^3, 2zx\right\rangle.\end{aligned}$$

(b) Note that $f(1, 1, 1) = 3$. Use your answer from part (a) to find the equation of the tangent plane to the surface $x^2 + y^4 + z^2x = 3$ at the point $(1, 1, 1)$.

The equation of the tangent plane to the level surface $f(x, y, z) = 3$ at the point $(1, 1, 1)$ is

$$\begin{aligned}\nabla f(1, 1, 1) \bullet \langle x - 1, y - 1, z - 1\rangle &= 0 \\ \langle 2(1) + (1)^2, 3(1)^3, 2(1)(1)\rangle \bullet \langle x - 1, y - 1, z - 1\rangle &= 0 \\ \langle 3, 4, 2\rangle \bullet \langle x - 1, y - 1, z - 1\rangle &= 0 \\ 3(x - 1) + 4(y - 1) + 2(z - 1) &= 0 \\ 3x + 4y + 2z &= 9.\end{aligned}$$

Here is a picture:



Problem 5. The function $f(x, y) = \sin(x)\sin(y)$ has infinitely many critical points. Here are three of them:

$$(0,0), \quad (\pi/2, \pi/2), \quad (\pi/2, -\pi/2).$$

(a) Compute the Hessian matrix of $f(x, y)$ and its determinant.

First we compute all first and second derivatives:

$$\begin{aligned} f_x &= \cos(x)\sin(y), \\ f_y &= \sin(x)\cos(y), \\ f_{xx} &= -\sin(x)\sin(y), \\ f_{yy} &= -\sin(x)\sin(y), \\ f_{xy} &= \cos(x)\cos(y), \\ f_{yx} &= \cos(x)\cos(y). \end{aligned}$$

Hence the Hessian determinant is

$$\begin{aligned} \det(Hf) &= \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \\ &= \det \begin{pmatrix} -\sin(x)\sin(y) & \cos(x)\cos(y) \\ \cos(x)\cos(y) & -\sin(x)\sin(y) \end{pmatrix} \\ &= [\sin(x)\sin(y)]^2 - [\cos(x)\cos(y)]. \end{aligned}$$

(b) Use the second derivative test to determine whether each of the three critical points listed above is a local max, local min or a saddle point.

The critical point $(0,0)$ satisfies $\det(Hf)(0,0) = [\sin(0)^2]^2 - [\cos(0)^2]^2 = 0 - 1 = -1 < 0$, so this is a saddle point. The critical point $(\pi/2, \pi/2)$ satisfies

$$\det(Hf)(\pi/2, \pi/2) = [\sin(\pi/2)^2]^2 - [\cos(\pi/2)^2]^2 = 1^2 - 0^2 = 1 > 0,$$

so this is a local max or min. Since $f_{xx}(\pi/2, \pi/2) = -\sin(\pi/2)^2 = -1 < 0$ it is a local max. The critical point $(\pi/2, -\pi/2)$ satisfies

$$\begin{aligned}\det(Hf)(\pi/2, -\pi/2) &= [\sin(\pi/2)\sin(-\pi/2)]^2 - [\cos(\pi/2)\cos(-\pi/2)]^2 \\ &= [(1)(-1)]^2 - [0]^2 \\ &= 1 < 0,\end{aligned}$$

so this is a local max or min. Since $f_{xx}(\pi/2, -\pi/2) = -\sin(\pi/2)\sin(-\pi/2) = -(1)(-1) = 1 > 0$ it is a local min. Here is a picture: