Math 310 **Fall 2023**
Homework 6 **Drew Armstrong**

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1. Various Kinds of First and Second Derivatives in $\mathbb{R}^3$. For any scalar field $f(x,y,z)$ we define a vector field grad(f) and a scalar field laplacian(f) by

$$\text{grad}(f) = \text{"}\nabla f\text{"} = \langle f_x, f_y, f_z\rangle,$$
$$\text{laplacian}(f) = \text{"}\nabla^2 f\text{"} = f_{xx} + f_{yy} + f_{zz}.$$

and for any vector field $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ we define a vector field curl$(\mathbf{F})$ and a scalar field div$(\mathbf{F})$ by

$$\text{curl}(\mathbf{F}) = \text{"}\nabla \times \mathbf{F}\text{"} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle,$$
$$\text{div}(\mathbf{F}) = \text{"}\nabla \bullet \mathbf{F}\text{"} = P_x + Q_y + R_z.$$

(a) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that curl(grad(f)) $= \langle 0,0,0\rangle$.
(b) For any vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ check that div(curl$(\mathbf{F})$) $= 0$.
(c) For any scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ check that div(grad(f)) $=$ laplacian(f).

2. Conservative Vector Fields. Consider the vector field

$$\mathbf{F}(x,y,z) = \langle y+z, x+z, x+y\rangle .$$

(a) Check that the curl is constantly zero: $\nabla \times \mathbf{F}(x,y,z) = \langle 0,0,0\rangle$.
(b) It follows from part (a) that there exists some scalar field $f(x,y,z)$ satisfying $\nabla f = \mathbf{F}$. Find one such scalar field. [Hint: Integrate $\mathbf{F}$ along an arbitrary path starting at some arbitrary point and ending at (x,y,z).]

3. Gravitational Potential. A sun of mass M sits at the origin in $\mathbb{R}^3$. According to Newton, the gravitational force due to the sun acting on a particle of mass m at the point (x,y,z) has the form

$$\mathbf{F}(x,y,z) = \frac{-GMm}{(x^2+y^2+z^2)^{3/2}} \cdot \langle x,y,z\rangle,$$

where G is the gravitational constant.

(a) Check that the following scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies $\nabla f = \mathbf{F}$:

$$f(x,y,z) = \frac{+GMm}{\sqrt{x^2+y^2+z^2}}.$$

(b) We can think of $U = -f$ as the *gravitational potential energy*. Suppose that our particle is held at rest at a position with distance D from the origin. At time zero the particle is allowed to fall towards the sun. If the sun has radius R, use conservation of energy to compute the particle's speed when it hits the sun's surface. Your answer will involve the constants G, M, R and D (but not m). [Assume that $D > R$.]

4. Line Integrals and Flux Integrals in $\mathbb{R}^2$. Let C be the parametrized path $\mathbf{r}(t) = (t, t^2)$ for $0 \le t \le 1$, with velocity vector $\mathbf{r}'(t) = \langle 1, 2t\rangle$. From this parametrization we can define a unit tangent vector and a unit normal vector to the curve C at the point $\mathbf{r}(t)$:

$$\mathbf{T} = \frac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}} \quad \text{and} \quad \mathbf{N} = \frac{\langle 2t, -1\rangle}{\sqrt{1+4t^2}}.$$

Now consider the constant vector field $\mathbf{F}(x,y) = \langle 3, 1\rangle$.

(a) Compute the line integral $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$.

(b) Compute the flux integral $\int_C \mathbf{F} \bullet \mathbf{N}\, ds$.

5. Green's Theorem on a Circle. Let D be the unit disk in $\mathbb{R}^2$ centered at $(0,0)$. Consider the vector field $\mathbf{F} = \langle P, Q\rangle = \langle xy^2, x+y\rangle$ with $\operatorname{curl}(\mathbf{F}) = Q_x - P_y = 1 - 2xy$.

(a) Compute the integral $\iint_D \operatorname{curl}(\mathbf{F})\, dA$. [Hint: Polar coordinates are easiest. You may use the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$.]

(b) The boundary curve ∂D is the unit circle, oriented counterclockwise. Use the standard parametrization $\mathbf{r}(t) = (\cos t, \sin t)$ with $0 \le t \le 2\pi$ to set up the integral $\oint_{\partial D} \mathbf{F} \bullet \mathbf{T}\, ds$. You will probably not be able to evaluate this integral by hand. Use a computer to verify that you get the same answer as in part (a).

6. Stokes' Theorem on a Parabolic Dome. Let D be the two-dimensional surface in $\mathbb{R}^3$ defined by $z = 1 - x^2 - y^2$ and $z \ge 0$. This surface can be parametrized by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 1-u^2\rangle \quad \text{with } 0 \le u \le 1 \text{ and } 0 \le v \le 2\pi.$$

The boundary curve ∂D is the unit circle in the x,y-plane, oriented counterclockwise, which can be parametrized as $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ for $0 \le t \le 2\pi$. Consider the vector field $\mathbf{F}(x,y,z) = \langle z, x, y\rangle$, which has constant curl vector $\nabla \times \mathbf{F} = \langle 1,1,1\rangle$.

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and their cross product $\mathbf{r}_u \times \mathbf{r}_v$.

(b) Use part (a) to compute the flux of the vector field $\nabla \times \mathbf{F}$ across the surface D:

$$\iint_D (\nabla \times \mathbf{F}) \bullet \mathbf{N}\, dA = \iint_D (\nabla \times \mathbf{F})(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v))\, du dv.$$

(c) Now compute the circulation of the vector field $\mathbf{F}$ around the boundary curve ∂D:

$$\int_{\partial D} \mathbf{F} \bullet \mathbf{T}\, ds = \int_{\partial D} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt.$$

Make sure that you get the same answer as in part (a).