Math 310 **Fall 2023**
Homework 5 **Drew Armstrong**

1. Integral Along a Parabola. Integrate the scalar function $f(x, y) = x$ along the parametrized curve $\mathbf{r}(t) = (t, t^2)$ for $0 \leq t \leq 1$.

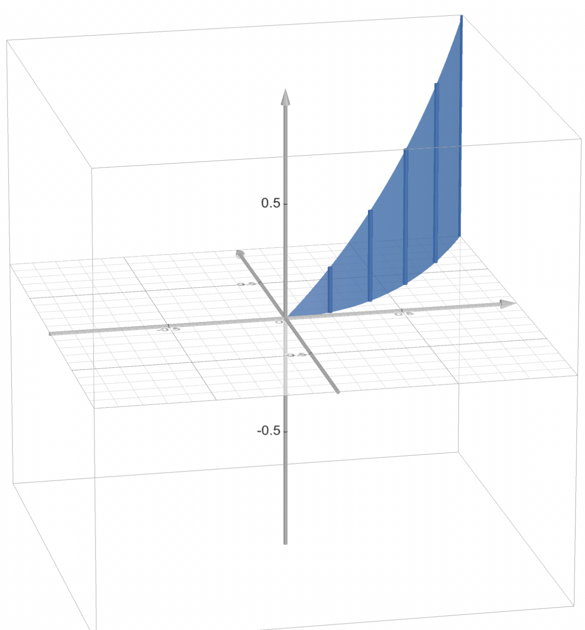
The velocity vector is $\mathbf{r}'(t) = \langle 1, 2t \rangle$ and the value of the function $f(x, y) = x$ at the point $\mathbf{r}(t) = (x(t), y(t)) = (t, t^2)$ is

$$f(\mathbf{r}(t)) = f(x(t), y(t)) = x(t) = t.$$

Hence the integral is

$$\begin{aligned}
\int_{\text{curve}} f &= \int_0^1 f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt \\
&= \int_0^1 t\sqrt{1^2 + (2t)^2} \, dt \\
&= \int_0^1 t\sqrt{1 + 4t^2} \, dt \\
&= \frac{1}{8} \int_{u=1}^{u=5} \sqrt{u} \, du \qquad\qquad (u = 1 + 4t^2, du = 8t \, dt) \\
&= \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_1^5 \\
&= \frac{1}{12} \left(5^{3/2} - 1^{3/2} \right) \approx 0.84.
\end{aligned}$$

Remark: We can think of this as the area of a wall whose base is a the parabola (t, t^2) for $0 \leq t \leq 1$, and whose height above the point (t, t^2) is t:[1]



[1] `https://www.desmos.com/3d/dda5d91337`

2. Projection. Let $\mathbf{F}$ and $\mathbf{u}$ be any vectors in $\mathbb{R}^n$ with $\|\mathbf{u}\| = 1$.

(a) The *component of* $\mathbf{F}$ *in the direction of* $\mathbf{u}$ has the form $t\mathbf{u}$ for some scalar t. Prove that $t = \mathbf{F} \bullet \mathbf{u}$. [Hint: This is much easier than it looks. We assume that the vector $\mathbf{F} - t\mathbf{u}$ is perpendicular to $\mathbf{u}$ so their dot product is zero: $(\mathbf{F} - t\mathbf{u}) \bullet \mathbf{u} = 0$. Solve for t.]

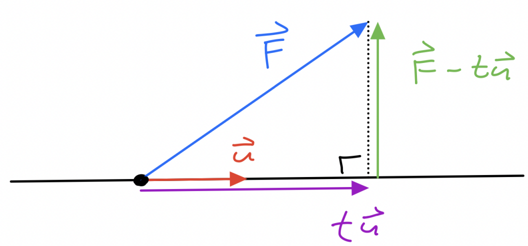
(b) Draw a picture of the three vectors $\mathbf{F}$, $\mathbf{u}$ and $(\mathbf{F} \bullet \mathbf{u})\mathbf{u}$.

(a): We have assumed that $\|\mathbf{u}\| = 1$, so that $\mathbf{u} \bullet \mathbf{u} = \|\mathbf{u}\|^2 = 1$. And we have assumed that $\mathbf{F} - t\mathbf{u}$ and $\mathbf{u}$ are perpendicular, so that

$$\begin{aligned}
(\mathbf{F} - t\mathbf{u}) \bullet \mathbf{u} &= 0 \\
\mathbf{F} \bullet \mathbf{u} - t\mathbf{u} \bullet \mathbf{u} &= 0 \\
\mathbf{F} \bullet \mathbf{u} &= t\mathbf{u} \bullet \mathbf{u} \\
\mathbf{F} \bullet \mathbf{u} &= t.
\end{aligned}$$

It follows that the projection of $\mathbf{F}$ onto $\mathbf{u}$ is $t\mathbf{u} = (\mathbf{F} \bullet \mathbf{u})\mathbf{u}$.

(b): There are many ways to picture this. Here is one way:



Remark: It we let θ denote the angle between $\mathbf{u}$ and $\mathbf{F}$, measured tail-to-tail, then we can also express the projection as $t\mathbf{u}$ where $t = \mathbf{F} \bullet \mathbf{u} = \|\mathbf{F}\|\|\mathbf{u}\| \cos\theta = \|\mathbf{F}\| \cos\theta$. However, angles are awkward to work with, so we prefer to express the projection in terms of the dot product.

3. Area a Parallelogram. For any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^3$ prove that

$$\|\mathbf{x} \times \mathbf{y}\| = \sqrt{\det \begin{pmatrix} \mathbf{x} \bullet \mathbf{x} & \mathbf{x} \bullet \mathbf{y} \\ \mathbf{x} \bullet \mathbf{y} & \mathbf{y} \bullet \mathbf{y} \end{pmatrix}}$$

[Hint: Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$, measured tail-to-tail. From a previous chapter we know that $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\theta$ and $\mathbf{x} \bullet \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$.]

Let θ be the angle between vectors $\mathbf{x}$ and $\mathbf{y}$, measured tail-to-tail. We saw in a previous chapter that $\mathbf{x} \bullet \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$ and $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\theta$. Hence we have

$$\begin{aligned}
\det \begin{pmatrix} \mathbf{x} \bullet \mathbf{x} & \mathbf{x} \bullet \mathbf{y} \\ \mathbf{x} \bullet \mathbf{y} & \mathbf{y} \bullet \mathbf{y} \end{pmatrix} &= (\mathbf{x} \bullet \mathbf{x})(\mathbf{y} \bullet \mathbf{y}) - (\mathbf{x} \bullet \mathbf{y})^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\|\mathbf{x}\|\|\mathbf{y}\| \cos\theta)^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \cos^2\theta \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 (1 - \cos^2\theta) \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \sin^2\theta \\
&= (\|\mathbf{x}\|\|\mathbf{y}\| \sin\theta)^2 \\
&= \|\mathbf{x} \times \mathbf{y}\|^2.
\end{aligned}$$

Then taking the square root of each side gives the result.

Remark: The factor $\|\mathbf{r}_u \times \mathbf{r}_v\|$ appears in the usual definition of the integral over a 2D surface living in 3D. But the cross product only exists in 3D. By using the determinant formula above we can define the integral of a scalar function f over a two-dimensional parametrized surface $\mathbf{r}(u,v)$ living in higher-dimensional space:

$$\iint_{\text{surface}} f = \iint f(\mathbf{r}(u,v))\sqrt{\det\begin{pmatrix}\mathbf{r}_u \bullet \mathbf{r}_u & \mathbf{r}_u \bullet \mathbf{r}_v \\ \mathbf{r}_v \bullet \mathbf{r}_u & \mathbf{r}_v \bullet \mathbf{r}_v\end{pmatrix}}\, dudv.$$

Even in three dimensions, this formula is sometimes easier than the cross product formula.

4. A Parametrized Torus. Fix two radii $a > b > 0$ and consider the parametrized torus

$$\begin{aligned}\mathbf{r}(u,v) &= \langle x(u,v), y(u,v), z(u,v)\rangle \\ &= \langle (a + b\cos(u))\cos(v), (a + b\cos(u))\sin(v), b\sin(u)\rangle,\end{aligned}$$

with $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$.

(a) Compute the tangent vectors $\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$.

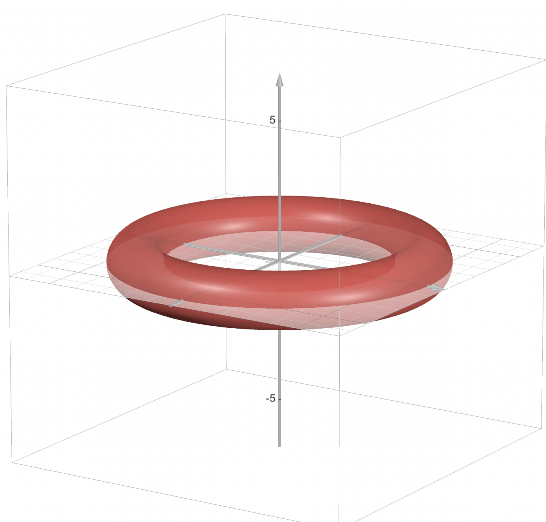
(b) Use your answer from part (a) to show that

$$\begin{aligned}\mathbf{r}_u \bullet \mathbf{r}_u &= b^2, \\ \mathbf{r}_v \bullet \mathbf{r}_v &= (a + b\cos(u))^2, \\ \mathbf{r}_u \bullet \mathbf{r}_v &= 0.\end{aligned}$$

(c) Use part (b) and Problem 3 to show that $\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b\cos(u))$.

(d) Use part (c) to compute the surface area of the torus: $\iint 1 \cdot \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv$.

Remark: Here is a picture of the torus with $a = 5$ and $b = 1$:[2]



(a): We have

$$\begin{aligned}\mathbf{r}_u &= \langle -b\sin u\cos v, -b\sin u\sin v, b\cos u\rangle, \\ \mathbf{r}_v &= \langle -(a + b\cos u)\sin v, (a + b\cos u)\cos v, 0\rangle.\end{aligned}$$

[2]`https://www.desmos.com/3d/4c95305123`

(b): We have

$$\begin{aligned}
\mathbf{r}_u \bullet \mathbf{r}_u &= (-b\sin u\cos v)^2 + (-b\sin u\sin v)^2 + (b\cos u)^2 \\
&= b^2\sin^2 u\cos^2 v + b^2\sin^2 u\sin^2 v + b^2\cos^2 u \\
&= b^2\sin^2 u\cancel{(\cos^2 v + \sin^2 v)} + b^2\cos^2 u \\
&= b^2\cancel{(\sin^2 u + \cos^2 u)} \\
&= b^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}_v \bullet \mathbf{r}_v &= (-(a+b\cos u)\sin v)^2 + ((a+b\cos u)\cos v)^2 + 0^2 \\
&= (a+b\cos u)^2\sin^2 v + (a+b\cos u)^2\cos^2 v \\
&= (a+b\cos u)^2\cancel{(\sin^2 v + \cos^2 v)} \\
&= (a+b\cos u)^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}_u \bullet \mathbf{r}_v &= (-b\sin u\cos v)(-(a+b\cos u)\sin v) + (-b\sin u\sin v)((a+b\cos u)\cos v) + 0 \\
&= b\sin u\cos v\sin v(a+b\cos u) - b\sin u\cos v\sin v(a+b\cos u) \\
&= 0.
\end{aligned}$$

(c): It follows from part (b) and Problem 3 that

$$\begin{aligned}
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{\det\begin{pmatrix} \mathbf{r}_u \bullet \mathbf{r}_u & \mathbf{r}_u \bullet \mathbf{r}_v \\ \mathbf{r}_u \bullet \mathbf{r}_v & \mathbf{r}_v \bullet \mathbf{r}_v \end{pmatrix}} \\
&= \sqrt{\det\begin{pmatrix} b^2 & 0 \\ 0 & (a+b\cos u)^2 \end{pmatrix}} \\
&= \sqrt{b^2(a+b\cos u)^2 - (0)(0)} \\
&= b(a+b\cos u).
\end{aligned}$$

This expression is always positive because we have assumed that $a > b > 0$.

(d): Recall that the surface of the torus is parametrized by $\mathbf{r}(u,v)$ with $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Hence the surface area of the torus is

$$\begin{aligned}
\iint_{\text{torus}} 1 &= \iint 1 \cdot \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv \\
&= \iint b(a+b\cos u)\, dudv \\
&= b\int_0^{2\pi} 1\, dv \int_0^{2\pi} (a+b\cos u)\, du \\
&= b(2\pi)\left[au - b\sin u\right]_0^{2\pi} \\
&= b(2\pi)\left(2\pi a - b\sin(2\pi) - a(0) + b\sin(0)\right) \\
&= 4\pi^2 ab.
\end{aligned}$$

Remark: A rectangle with side lengths $2\pi a$ and $2\pi b$ has area $(2\pi a)(2\pi b) = 4\pi^2 ab$. We can turn this rectangle into a torus by first rolling the rectangle into a tube of radius b and length

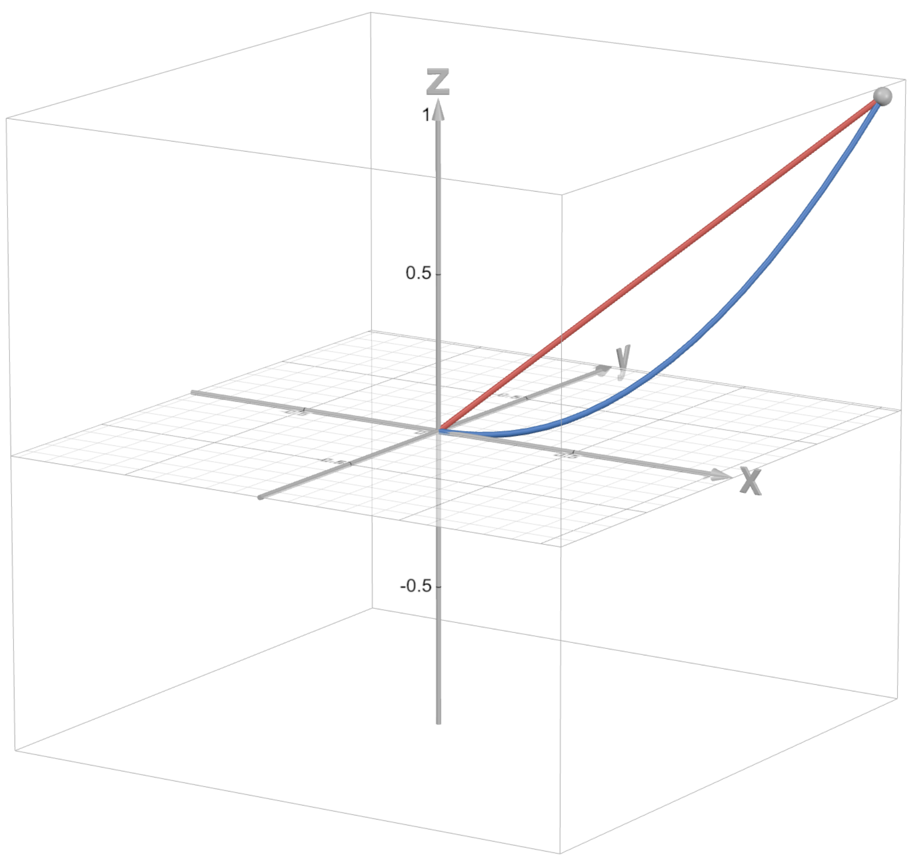
$2\pi a$, then curving the tube into a torus. The first operation does not affect the area. The second operation **does** affect the area, **however** the squishing on the inside of the torus and the stretching on the outside of the torus exactly cancel, so the torus has the same surface area as the original rectangle. [We can avoid stretching by working in four-dimensional space.]

5. A Conservative Vector Field. Consider the scalar function $f(x, y, z) = xyz + 7$ and its gradient vector field $\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle yz, xz, xy \rangle$. Recall that the integral of a vector field $\mathbf{F}$ along a parametrized curve $\mathbf{r}(t)$ is defined as follows:

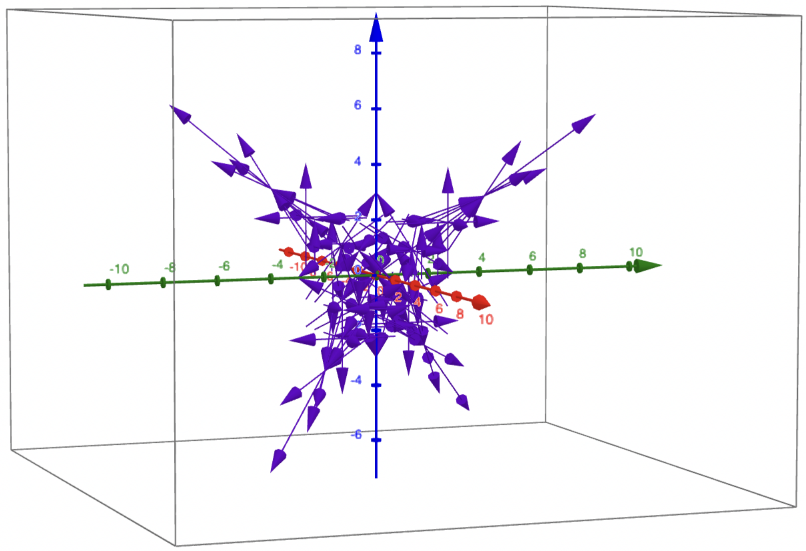
$$\int_{\text{Curve}} \mathbf{F} = \int \left(\mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \right) \|\mathbf{r}'(t)\| \, dt = \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt.$$

(a) Compute the integral of $\mathbf{F}$ along the curve $\mathbf{r}(t) = (t, t, t)$ for $0 \le t \le 1$.
(b) Compute the integral of $\mathbf{F}$ along the curve $\mathbf{r}(t) = (t, t^2, t^3)$ for $0 \le t \le 1$.
(c) Compute $f(1, 1, 1) - f(0, 0, 0)$.

Remark: Here is a picture of the two paths from parts (a) and (b):



The vector field $\mathbf{F} = \langle yz, xz, xy \rangle$ is quite difficult to visualize. Here is a very rough picture:



Note that the vector field is pushing away from the origin in four different directions.

(a): If $\mathbf{r}(t) = (t, t, t)$ then we have $\mathbf{r}'(t) = \langle 1, 1, 1 \rangle$, and the integral of $\mathbf{F}$ along $\mathbf{r}$ is

$$\begin{aligned}
\int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int_0^1 \mathbf{F}(t, t, t) \bullet \langle 1, 1, 1 \rangle\,dt \\
&= \int_0^1 \langle t^2, t^2, t^2 \rangle \bullet \langle 1, 1, 1 \rangle\,dt \\
&= \int_0^1 (t^2 + t^2 + t^2)\,dt \\
&= \int_0^1 3t^2\,dt \\
&= 3\frac{t^3}{3}\bigg|_0^1 \\
&= 1.
\end{aligned}$$

(b): If $\mathbf{r}(t) = (t, t^2, t^3)$ then we have $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$, and the integral of $\mathbf{F}$ along $\mathbf{r}$ is

$$\begin{aligned}
\int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int_0^1 \mathbf{F}(t, t^2, t^3) \bullet \langle 1, 2t, 3t^2 \rangle\,dt \\
&= \int_0^1 \langle t^5, t^4, t^3 \rangle \bullet \langle 1, 2t, 3t^2 \rangle\,dt \\
&= \int_0^1 (t^5 + 2t^5 + 3t^5)\,dt \\
&= \int_0^1 6t^5\,dt \\
&= 6\frac{t^6}{6}\bigg|_0^1 \\
&= 1.
\end{aligned}$$

(c): Note that $f(1,1,1) - f(0,0,0) = (1 \cdot 1 \cdot 1 + 7) - (0 \cdot 0 \cdot 0 + 7) = 8 - 7 = 1$.

Remark: The answers from parts (a), (b), (c) are the same because of the Fundamental Theorem of Line Integrals, which says that for any scalar field f and for any path $\mathbf{r}(t)$ we have

$$\int_0^1 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = f(\mathbf{r}(1)) - f(\mathbf{r}(0)).$$

6. Circulation of Vector Fields. Consider the vector fields $\mathbf{F} = \langle -y, x\rangle$ and $\mathbf{G} = \langle x, y\rangle$.

(a) Compute the integral of $\mathbf{F}$ around the circle $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ and observe that the result is **not** equal to zero. It follows from this that $\mathbf{F}$ cannot be expressed in the form $\mathbf{F} = \nabla f$ for any scalar function $f(x, y)$.

(b) Compute the integral of $\mathbf{G}$ around the circle $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ and observe that the result **is** equal to zero.

(c) In fact, it is true that the integral of $\mathbf{G}$ around any closed loop is zero, which implies that $\mathbf{G} = \nabla g$ for some scalar function $g(x, y)$. Find one such function. [Hint: You could just guess, but there is a systematic method based on the Fundamental Theorem of Line Integrals:

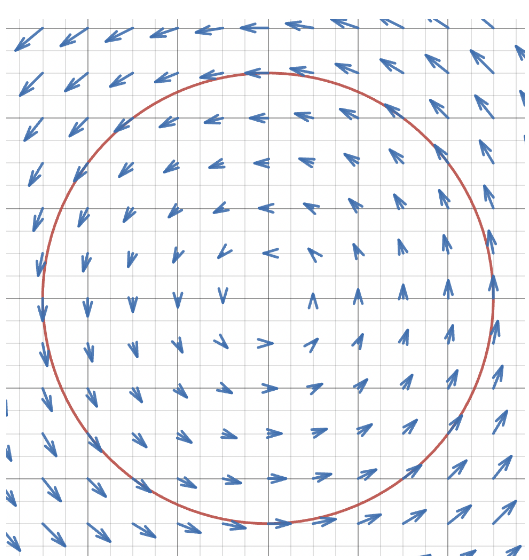
$$\int_0^1 \nabla g(\mathbf{r}(t)) \bullet \mathbf{r}'(t), dt = g(\mathbf{r}(1)) - g(\mathbf{r}(0)).$$

The path $\mathbf{r}(t) = (xt, yt)$ has $\mathbf{r}(1) = (x, y)$. Compute the function

$$g(x, y) := \int_0^1 \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_0^1 \mathbf{G}(xt, yt) \bullet \langle x, y\rangle\, dt$$

and check that this function satisfies $\nabla g = \mathbf{G}$.]

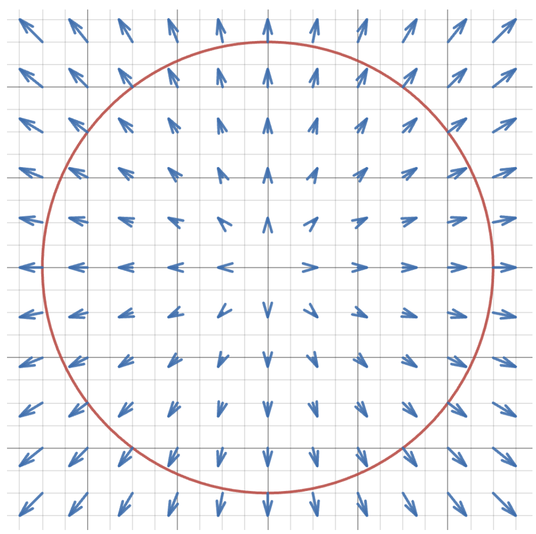
(a): We will integrate the vector field $\mathbf{F} = \langle -y, x\rangle$ around a counterclockwise circle $\mathbf{r}(t) = (\cos t, \sin t)$. Here is a picture:



Since the vector field is always pointing in the same direction as the velocity $\mathbf{r}'(t)$ we expect that the answer will be a positive number. Indeed, we have

$$\begin{aligned}\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt &= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \bullet \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} ((-\sin t)^2 + (\cos t)^2)\, dt \\ &= \int_0^{2\pi} 1\, dt \\ &= 2\pi.\end{aligned}$$

(b): We will integrate the vector field $\mathbf{G} = \langle x, y\rangle$ around a counterclockwise circle $\mathbf{r}(t) = (\cos t, \sin t)$. Here is a picture:



Since the vector field $\mathbf{G}$ is always perpendicular to the velocity $\mathbf{r}'(t)$, we are just integrating the number zero over the curve:

$$\begin{aligned}\int_0^{2\pi} \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt &= \int_0^{2\pi} \mathbf{G}(\cos t, \sin t) \bullet \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} \langle \cos t, \sin t\rangle \bullet \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)\, dt \\ &= \int_0^{2\pi} 0\, dt \\ &= 0.\end{aligned}$$

(c): Since the integral of $\mathbf{F}$ around a closed loop is not zero, the Fundamental Theorem of Line Integrals tells us that there does **not** exist any scalar function f such that $\mathbf{F} = \nabla f$. But

I claim that there does exist a scalar function g satisfying $\mathbf{G} = \nabla g$. This function must satisfy

$$\nabla g = \langle \partial g/\partial x, \partial g/\partial y \rangle = \langle x, y \rangle,$$

so that $\partial g/\partial x = x$ and $\partial g/\partial y = y$. Can you think of such a function?

You could try an ad hoc method, but here is a systematic method. We define $g(x,y)$ as the integral of $\mathbf{G}$ over the path $\mathbf{r}(t) = (xt, yt)$ for $0 \leq t \leq 1$:

$$\begin{aligned}
g(x,y) &:= \int_0^1 \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^1 \mathbf{G}(xt, yt) \bullet \langle x, y \rangle\, dt \\
&= \int_0^1 \langle xt, yt \rangle \bullet \langle x, y \rangle\, dt \\
&= \int_0^1 (x^2 t + y^2 t)\, dt \\
&= \left[x^2 \frac{t^2}{2} + y^2 \frac{t^2}{2} \right]_0^1 \\
&= (x^2 + y^2)/2.
\end{aligned}$$

We verify that

$$\nabla \left(\frac{x^2 + y^2}{2} \right) = \left\langle \frac{2x + 0}{2}, \frac{0 + 2y}{2} \right\rangle = \langle x, y \rangle.$$

Remark: This method works because of the Fundamental Theorem of Line Integrals. In Calculus I you learned that for any function $G : \mathbb{R} \to \mathbb{R}$, the function

$$g(x) := \int_{\text{anything}}^{x} G(t)\, dt$$

satisfies $g'(x) = G(x)$. The lower limit is arbitrary. We can use a similar trick for line integrals of vector fields. Roughly, we define

$$g(x,y) := \int_{\text{any point}}^{(x,y)} \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt,$$

where the integral is taken over **any path** $\mathbf{r}(t)$ with endpoint (x, y). If $\mathbf{G}$ is a conservative[3] vector field then this integral is independent of the shape of the path, and one can use the Fundamental Theorem of Line Integrals to show that $\nabla g = \mathbf{G}$. In order to use this trick, I chose the simplest possible path with endpoint (x, y); namely, a straight line from $(0, 0)$ to (x, y). Changing the starting point would only add a scalar to g, which does not change ∇g.

[3] We will discuss this concept in detail next week.