Math 310 **Fall 2023**
Homework 5 **Drew Armstrong**

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1. Integral Along a Parabola. Integrate the scalar function $f(x, y) = x$ along the parametrized curve $\mathbf{r}(t) = (t, t^2)$ for $0 \le t \le 1$.

2. Projection. Let $\mathbf{F}$ and $\mathbf{u}$ be any vectors in $\mathbb{R}^n$ with $\|\mathbf{u}\| = 1$.

(a) The *component of* $\mathbf{F}$ *in the direction of* $\mathbf{u}$ has the form $t\mathbf{u}$ for some scalar t. Prove that $t = \mathbf{F} \bullet \mathbf{u}$. [Hint: This is much easier than it looks. We assume that the vector $\mathbf{F} - t\mathbf{u}$ is perpendicular to $\mathbf{u}$ so their dot product is zero: $(\mathbf{F} - t\mathbf{u}) \bullet \mathbf{u} = 0$. Solve for t.]

(b) Draw a picture of the three vectors $\mathbf{F}$, $\mathbf{u}$ and $(\mathbf{F} \bullet \mathbf{u})\mathbf{u}$.

3. Area a Parallelogram. For any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^3$ prove that

$$\|\mathbf{x} \times \mathbf{y}\| = \sqrt{\det \begin{pmatrix} \mathbf{x} \bullet \mathbf{x} & \mathbf{x} \bullet \mathbf{y} \\ \mathbf{x} \bullet \mathbf{y} & \mathbf{y} \bullet \mathbf{y} \end{pmatrix}}$$

[Hint: Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$, measured tail-to-tail. From a previous chapter we know that $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\theta$ and $\mathbf{x} \bullet \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$.]

4. A Parametrized Torus. Fix two radii $a > b > 0$ and consider the parametrized torus

$$\begin{aligned} \mathbf{r}(u, v) &= \langle x(u, v), y(u, v), z(u, v) \rangle \\ &= \langle (a + b\cos(u))\cos(v), (a + b\cos(u))\sin(v), b\sin(u) \rangle, \end{aligned}$$

with $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$.

(a) Compute the tangent vectors $\mathbf{r}_u = \langle x_u, y_u, z_u \rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v \rangle$.

(b) Use your answer from part (a) to show that

$$\begin{aligned} \mathbf{r}_u \bullet \mathbf{r}_u &= b^2, \\ \mathbf{r}_v \bullet \mathbf{r}_v &= (a + b\cos(u))^2, \\ \mathbf{r}_u \bullet \mathbf{r}_v &= 0. \end{aligned}$$

(c) Use part (b) and Problem 3 to show that $\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b\cos(u))$.

(d) Use part (c) to compute the surface area of the torus: $\iint 1 \cdot \|\mathbf{r}_u \times \mathbf{r}_v\| \, dudv$.

5. A Conservative Vector Field. Consider the scalar function $f(x, y, z) = xyz + 7$ and its gradient vector field $\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle yz, xz, xy \rangle$. Recall that the integral of a vector field $\mathbf{F}$ along a parametrized curve $\mathbf{r}(t)$ is defined as follows:

$$\int_{\text{Curve}} \mathbf{F} = \int \left(\mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \right) \|\mathbf{r}'(t)\| \, dt = \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \, dt.$$

(a) Compute the integral of $\mathbf{F}$ along the curve $\mathbf{r}(t) = (t, t, t)$ for $0 \le t \le 1$.

(b) Compute the integral of $\mathbf{F}$ along the curve $\mathbf{r}(t) = (t, t^2, t^3)$ for $0 \le t \le 1$.

(c) Compute $f(1, 1, 1) - f(0, 0, 0)$.

6. Circulation of Vector Fields. Consider the vector fields $\mathbf{F} = \langle -y, x \rangle$ and $\mathbf{G} = \langle x, y \rangle$.

(a) Compute the integral of $\mathbf{F}$ around the circle $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$ and observe that the result is **not** equal to zero. It follows from this that $\mathbf{F}$ cannot be expressed in the form $\mathbf{F} = \nabla f$ for any scalar function $f(x, y)$.

(b) Compute the integral of $\mathbf{G}$ around the circle $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$ and observe that the result **is** equal to zero.

(c) In fact, it is true that the integral of $\mathbf{G}$ around any closed loop is zero, which implies that $\mathbf{G} = \nabla g$ for some scalar function $g(x, y)$. Find one such function. [Hint: You could just guess, but there is a systematic method based on the Fundamental Theorem of Line Integrals:

$$\int_0^1 \nabla g(\mathbf{r}(t)) \bullet \mathbf{r}'(t), dt = g(\mathbf{r}(1)) - g(\mathbf{r}(0)).$$

The path $\mathbf{r}(t) = (xt, yt)$ has $\mathbf{r}(1) = (x, y)$. Compute the function

$$g(x, y) := \int_0^1 \mathbf{G}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_0^1 \mathbf{G}(xt, yt) \bullet \langle x, y \rangle\, dt$$

and check that this function satisfies $\nabla g = \mathbf{G}$.]