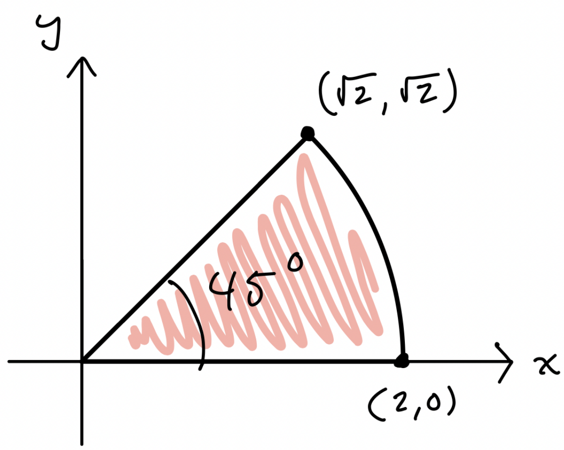
Problem 1. An Integral in the Plane. Consider the function $f(x,y)=x$. Let D be the region that is inside the circle $x^2+y^2=4$, above the line $y=0$ and below the line $y=x$.

(a) Draw the region. [Hint: It looks like 1/8 of a pie.]
(b) Compute the integral $\iint_D f(x,y)\,dxdy$ by converting to polar coordinates.
(c) Compute the integral $\iint_D f(x,y)\,dxdy$ in Cartesian coordinates by cutting the region D into two pieces D_1 and D_2 separated by the line $x=\sqrt{2}$. Check that you answers from parts (a) and (b) are the same.

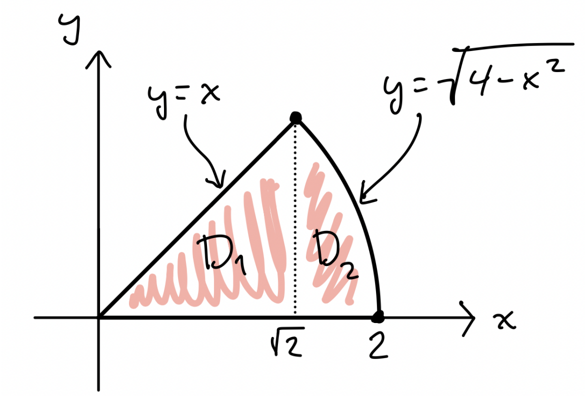
(a):



(b): The region D is parametrized in polar coordinates by $0\leq r\leq 2$ and $0\leq\theta\leq\pi/4$. Hence

$$\begin{aligned}
\iint_D x\,dxdy &= \iint_D x\,r dr d\theta \\
&= \iint_D r\cos\theta\,r dr d\theta \\
&= \iint_D r^2\cos\theta\,dr d\theta \\
&= \int_0^2 r^2\,dr\cdot\int_0^{\pi/4}\cos\theta\,d\theta \qquad \text{(separable)} \\
&= \left[\frac{r^3}{3}\right]_0^2\cdot[\sin\theta]_0^{\pi/4} \\
&= \left(\frac{2^3}{3}-\frac{0^3}{3}\right)\cdot(\sin(\pi/4)-\sin 0) \\
&= \frac{8}{3}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{4\sqrt{2}}{3}.
\end{aligned}$$

(c): We divide the region D into D_1 and D_2 as follows:



The region D_1 is parametrized by $0 \leq x \leq \sqrt{2}$ and $0 \leq y \leq x$, so that

$$\begin{aligned}
\iint_{D_1} x\, dxdy &= \int_0^{\sqrt{2}} \left(\int_0^x x\, dy \right) dx \\
&= \int_0^{\sqrt{2}} [xy]_{y=0}^{y=x}\, dx \\
&= \int_0^{\sqrt{2}} x^2\, dx \\
&= \left[\frac{x^3}{3} \right]_0^{\sqrt{2}} \\
&= \frac{2\sqrt{2}}{3}.
\end{aligned}$$

The region D_2 is parametrized by $\sqrt{2} \leq x \leq 2$ and $0 \leq y \leq \sqrt{4-x^2}$, so that

$$\begin{aligned}
\iint_{D_2} x\, dxdy &= \int_{\sqrt{2}}^2 \left(\int_0^{\sqrt{4-x^2}} x\, dy \right) dx \\
&= \int_{\sqrt{2}}^2 [xy]_{y=0}^{y=\sqrt{4-x^2}}\, dx \\
&= \int_{x=\sqrt{2}}^{x=2} x\sqrt{4-x^2}\, dx \\
&= \int_{u=0}^{u=2} -\frac{1}{2}\sqrt{u}\, du && (u = 4 - x^2, du = -2xdx) \\
&= \int_0^2 \frac{1}{2}\sqrt{u}\, du \\
&= \left[\frac{u^{3/2}}{3} \right]_0^2 \\
&= \frac{2\sqrt{2}}{3}.
\end{aligned}$$

We conclude that

$$\iint_D x\, dxdy = \iint_{D_1} x\, dxdy + \iint_{D_2} x\, dxdy = \frac{2\sqrt{2}}{3} + \frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{3},$$

which matches our answer from part (b).

[Remark: This problem illustrates the benefit of polar coordinates.]

Problem 2. Center of Mass. Let D be the same region as in Problem 1. Think of this as a thin metal plate with a constant density of 1 unit of mass per unit of area. Compute the following using polar coordinates.

(a) Compute the total mass $\iint_D 1\, dxdy$.
(b) Compute the moment about the y axis: $\iint_D x\, dxdy$.
(c) Compute the moment about the x axis: $\iint_D y\, dxdy$.
(d) Find the center of mass.

(a): As in Problem 1, we can parametrize this region in polar coordinates as $0 \le r \le 2$ and $0 \le \theta \le \pi/4$. The area is

$$\begin{aligned}
\text{area(D)} &= \iint_D 1\, dxdy \\
&= \iint_D 1\, r dr d\theta \\
&= \int_0^2 r\, dr \cdot \int_0^{\pi/4} 1\, d\theta \qquad \text{(separable)} \\
&= \left[\frac{r^2}{2}\right]_0^2 \cdot [\theta]_0^{\pi/4} \\
&= \left(\frac{2^2}{2} - \frac{0^2}{2}\right) \cdot \left(\frac{\pi}{4} - 0\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Indeed, our pie slice is 1/8 of a circle of radius 2, which has area $\pi(2)^2 = 4\pi$.

(b): We already computed this in Problem 1(b). The answer is $4\sqrt{2}/3$.

(c): Here we have

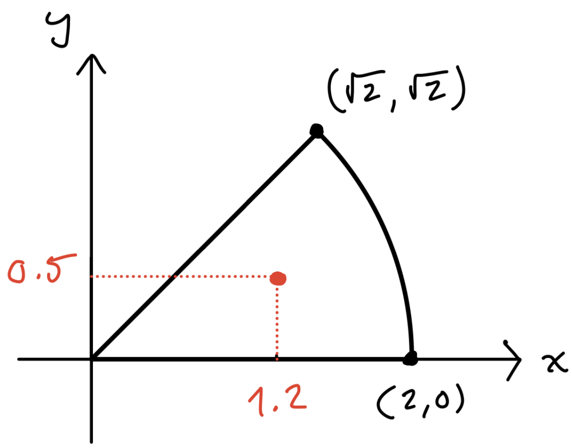
$$\begin{aligned}
\iint_D y\, dxdy &= \iint_D y\, r dr d\theta \\
&= \iint_D r \sin\theta\, r dr d\theta \\
&= \iint_D r^2 \sin\theta\, dr d\theta \\
&= \int_0^2 r^2\, dr \cdot \int_0^{\pi/4} \sin\theta\, d\theta \qquad \text{(separable)} \\
&= \left[\frac{r^3}{3}\right]_0^2 \cdot [-\cos\theta]_0^{\pi/4} \\
&= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) \cdot (-\cos(\pi/4) + \cos 0) \\
&= \frac{8}{3} \cdot \left(-\frac{\sqrt{2}}{2} + 1\right)
\end{aligned}$$

$$= \frac{4(2-\sqrt{2})}{3}.$$

(d): The center of mass is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\iint_D x}{\iint_D 1} = \frac{4\sqrt{2}/3}{\pi/2} = \frac{8\sqrt{2}}{3\pi} \quad \text{and} \quad \bar{y} = \frac{\iint_D y}{\iint_D 1} = \frac{4(2-\sqrt{2})/3}{\pi/2} = \frac{8(2-\sqrt{2})}{3\pi}.$$

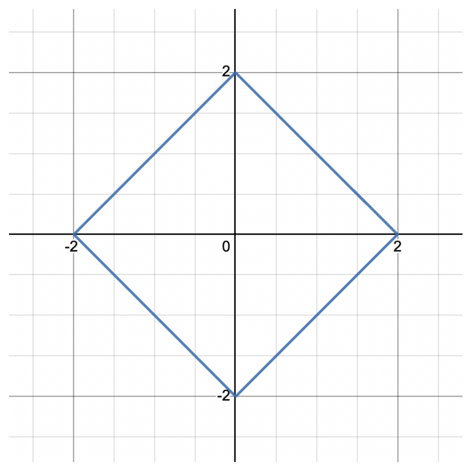
My computer says that $(\bar{x}, \bar{y}) \approx (1.2, 0.5)$. Here is a picture:



Problem 3. Change of Coordinates. Consider the function $f(x, y) = x^2 + y^2$. Let D be the square-shaped region in the x, y-plane bounded by the four lines $x + y = \pm 2$ and $x - y = \pm 2$.

(a) Draw the region.
(b) Consider the change of variables $x = u + v$ and $y = u - v$. Compute the area stretch factor (i.e., the absolute value of the determinant of the Jacobian matrix.)
(c) Compute the integral $\iint_D (x^2 + y^2)\, dxdy$ by converting to u, v-coordinates. [Hint: The region D in the u, v-plane is parametrized by $-1 \le u \le 1$ and $-1 \le v \le 1$.]

(a): Desmos produced the following picture:[1]



[1] https://www.desmos.com/calculator/jaxo0awiao

(b): This region is a little bit difficult to parametrize in Cartesian coordinates, so we change coordinates to $x = u + v$ and $y = u - v$. The area stretch factor is

$$\left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = |(1)(-1) - (1)(1)| = |-2| = 2.$$

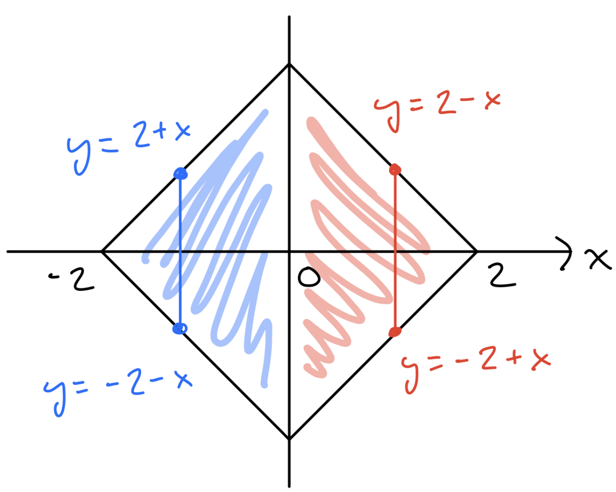
(c): From part (b) we know that $dxdy = 2dudv$. We also have

$$x^2 + y^2 = (u+v)^2 + (u-v)^2 = u^2 + 2uv + v^2 + u^2 - 2uv + v^2 = 2(u^2 + v^2).$$

Furthermore, we note that $x + y = (u+v) + (u-v) = 2u$ and $x - y = (u+v) - (u-v) = 2v$, so the region defined by $-2 \leq x + y \leq 2$ and $-2 \leq x - y \leq 2$ becomes $-2 \leq 2u \leq 2$ and $-2 \leq 2v \leq 2$, i.e., $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$. Thus we have

$$\begin{aligned}
\iint_D (x^2 + y^2)\, dxdy &= \iint_D 2(u^2 + v^2)\, 2dudv \\
&= 4\int_{-1}^{1} \left(\int_{-1}^{1} (u^2 + v^2)\, du\right) dv \\
&= 4\int_{-1}^{1} \left[\frac{u^3}{3} + v^2 u\right]_{u=-1}^{u=1} dv \\
&= 4\int_{-1}^{1} \left(\frac{2}{3} + 2v^2\right) dv \\
&= 4\left[\frac{2}{3}v + 2\frac{v^3}{3}\right]_{-1}^{1} \\
&= 4\left(\frac{4}{3} + \frac{4}{3}\right) \\
&= \frac{32}{3}.
\end{aligned}$$

Remark: One can check that this answer is correct by doing the more difficult computation in Cartesian coordinates. The sideways square D breaks into two triangles D_1 and D_2 where D_1 is parametrized by $-2 \leq x \leq 0$ and $-2 - x \leq y \leq 2 + x$ and where D_2 is parametrized by $0 \leq x \leq 2$ and $-2 + x \leq y \leq 2 - x$:

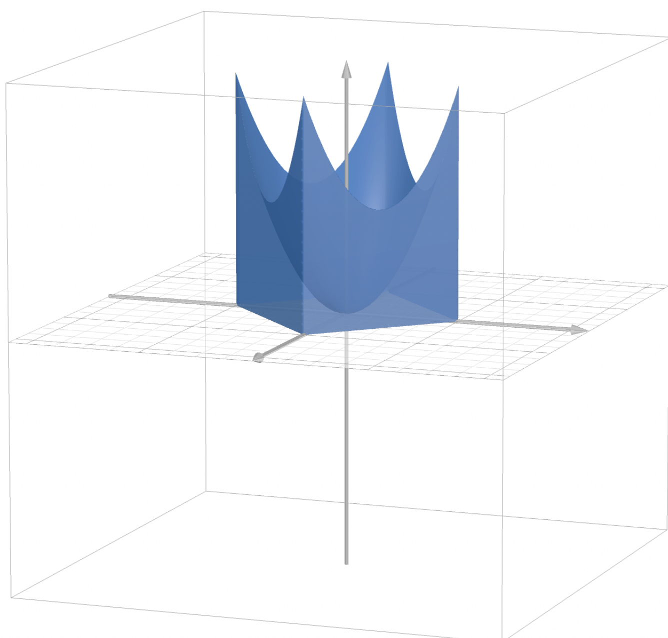


Then the integral over D is the sum of the integrals over D_1 and D_2. The integral over D_1 is

$$\begin{aligned}\iint_{D_1}(x^2+y^2)\,dxdy &= \int_{-2}^{0}\left(\int_{-2-x}^{2+x}(x^2+y^2)\,dy\right)\,dy \\ &= \int_{-2}^{0}\left[x^2y+\frac{y^3}{3}\right]_{y=-2-x}^{y=2+x}\,dx \\ &= \int_{-2}^{0}\left(x^2(2+x)+\frac{(2+x)^3}{3}-x^2(-2-x)-\frac{(-2-x)^3}{3}\right)\,dx,\end{aligned}$$

and **we do not want to compute this by hand**. I put it into my computer and it gave the answer 16/3. The integral over D_2 is also 16/3, so their sum is 32/3, as expected.

Remark: We can think of the number 32/3 as the **mass** of a thin plate D with density function x^2+y^2, so the corners of D are heavier than the center. Or we can think of 32/3 as the **volume** between the square D in the x,y-plane and the surface $z=x^2+y^2$ in x,y,z-space, which looks like a square-shaped crown:[2]



Problem 4. Integration Over a Rectangular Box. Let B be the rectangular box parametrized by $0\le x\le 1$, $0\le y\le 2$ and $0\le z\le 3$. Compute the triple integral

$$\iiint_B (x+y+z)\,dxdydz.$$

Since the limits of integration are constant we can perform the three integrals in any order. We will integrate over x,y,z in that order:

$$\int_0^3\left(\int_0^2\left(\int_0^1(x+y+z)\,dx\right)\,dy\right)\,dz$$

[2] `https://www.desmos.com/3d/1bcc69ab04`

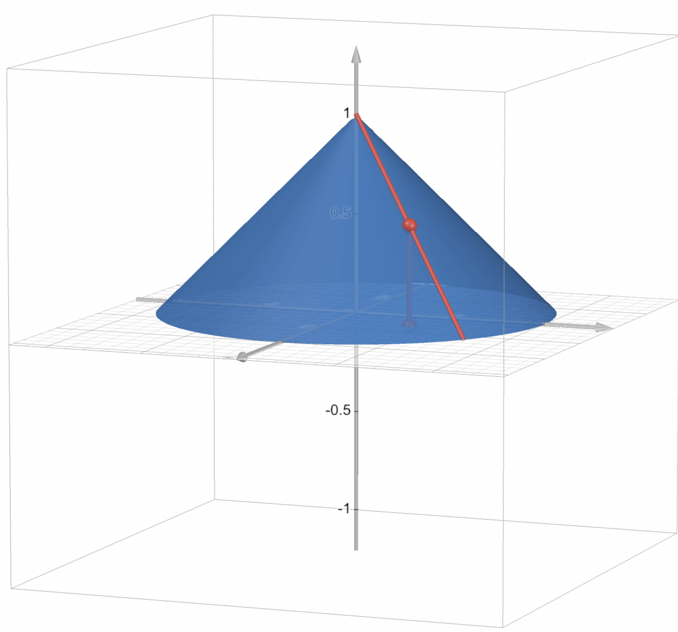
$$
\begin{aligned}
&= \int_0^3 \left(\int_0^2 \left[\frac{x^2}{2} + xy + xz \right]_{x=0}^{x=1} dy \right) dz \\
&= \int_0^3 \left(\int_0^2 \left(\frac{1}{2} + y + z \right) dy \right) dz \\
&= \int_0^3 \left[\frac{1}{2}y + \frac{y^2}{2} + yz \right]_{y=0}^{y=2} dz \\
&= \int_0^3 (3 + 2z)\, dz \\
&= \left[3z + z^2 \right]_{z=0}^{z=3} \\
&= 18.
\end{aligned}
$$

I don't have anything interesting to say about this.

Problem 5. Cylindrical Coordinates. Consider a solid cone of radius 1 and height 1 whose base is the unit disk $x^2 + y^2 \leq 1$ in the x, y-plane and whose vertex is at the point $(0, 0, 1)$ in x, y, z-space.

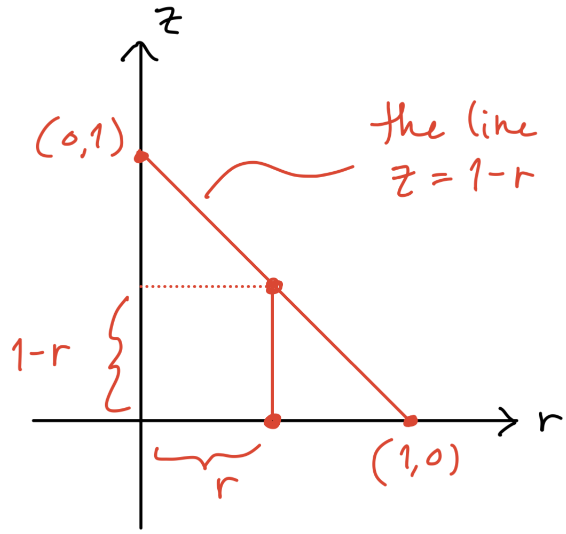
(a) Parametrize the cone using cylindrical coordinates: r, θ, z.
(b) Compute the volume of the cone.
(c) Compute the center of mass $(\bar{x}, \bar{y}, \bar{z})$, assuming that the cone has constant density 1. [Hint: By symmetry we know that $\bar{x} = 0$ and $\bar{y} = 0$, so you only have to compute $\bar{z}$.]

(a): We can parametrize the base circle by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Then for an arbitrary point in the base circle with coordinates r, θ we must determine the maximum value of the z-coordinate:[3]



[3] `https://www.desmos.com/3d/fe66f06a6f`

Since the cone is rotationally symmetric about the z-axis, the maximum value of z doesn't depend on θ. To examine the relationship between z and r, consider a vertical slice through the origin:



Note that the surface of the cone intersects this slice in the straight line $z = 1 - r$. Hence we should take $0 \le z \le 1 - r$.

(b): Now we can use cylindrical coordinates to compute the volume of the cone:

$$\begin{aligned}
\text{Vol(Cone)} &= \iiint_{\text{Cone}} 1\, dxdydz \\
&= \iiint_{\text{Cone}} r\, drd\theta dz && (dxdydz = rdrd\theta dz) \\
&= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 r \cdot \left(\int_0^{1-r} 1\, dz \right) dr && (\text{must integrate } z \text{ before } r) \\
&= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 r \cdot (1-r)\, dr \\
&= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 (r - r^2)\, dr \\
&= \int_0^{2\pi} 1\, d\theta \cdot \left[\frac{r^2}{2} - \frac{r^3}{3} \right]_0^1 \\
&= 2\pi \cdot \left(\frac{1}{2} - \frac{1}{3} \right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

Remark: The volume of a cone of radius r and height h is $\pi r^2 h/3$. When $r = 1$ and $h = 1$ this formula gives $\pi/3$, which agrees with our computation.

(c): By symmety we must have $\bar{x} = 0$ and $\bar{y} = 0$. The z coordinate of the center of mass is

$$\bar{z} = \iiint_{\text{Cone}} z\, dxdydz \Big/ \iiint_{\text{Cone}} 1\, dxdydz$$

$$
\begin{aligned}
&= \frac{3}{\pi} \cdot \iiint_{\text{Cone}} zr \, dr d\theta dz \\
&= \frac{3}{\pi} \cdot \int_0^{2\pi} 1 \, d\theta \cdot \int_0^1 r \cdot \left(\int_0^{1-r} z \, dz \right) \, dr \qquad\qquad \text{(must integrate } z \text{ before } r\text{)} \\
&= \frac{3}{\pi} \cdot \int_0^{2\pi} 1 \, d\theta \cdot \int_0^1 r \cdot (1-r)^2/2 \, dr \\
&= \frac{3}{2\pi} \cdot \int_0^{2\pi} 1 \, d\theta \cdot \int_0^1 (r^3 - 2r^2 + r) \, dr \\
&= \frac{3}{\pi} \cdot \int_0^{2\pi} 1 \, d\theta \cdot \left[\frac{r^4}{4} - 2\frac{r^3}{3} + \frac{r^2}{2} \right]_0^1 \\
&= \frac{3}{2\pi} \cdot 2\pi \cdot \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) \\
&= \frac{3}{2\pi} \cdot 2\pi \cdot \frac{1}{12} \\
&= \frac{1}{4}.
\end{aligned}
$$

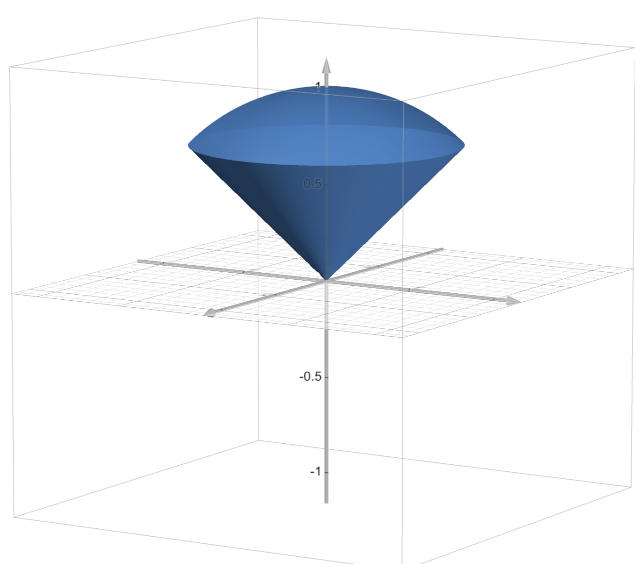
Hence the center of mass is $(0, 0, 1/4)$. Remark: We would get the same result for a cone of any radius and height. The center of mass is always exactly $1/4$ of the way from the center of the base to the apex.

Problem 6. Spherical Coordinates. Consider the "ice-cream-cone-shaped" solid region E that is between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z^2 = x^2 + y^2$, and satisfies $z \geq 0$. The volume is given by the triple integral:

$$
\text{Vol}(E) = \iiint_E 1 \, dx dy dz.
$$

Compute this integral by converting to spherical coordinates.

Here is a picture of the ice-cream cone, produced with Desmos:[4]



[4]`https://www.desmos.com/3d/8f0a6ad917`

This region is parametrized by $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/4$. (Note that the boundary of the cone has slope 1, hence angle $\pi/4$ from the vertical. One can see this by intersecting the cone $z^2 = x^2 + y^2$ with any vertical plane, such as $y = 0$, to get $z^2 = x^2$, and hence $z = \pm x$. This is a pair of lines of slope ± 1 in the x, z-plane.) Hence the volume is

$$\begin{aligned}
\mathrm{Vol(E)} &= \iiint_E 1\, dxdydz \\
&= \iiint_E \rho^2 \sin\varphi\, drd\theta dz && (dxdydz = \rho^2 \sin\varphi\, d\rho d\theta d\varphi) \\
&= \int_0^1 \rho^2\, d\rho \cdot \int_0^{2\pi} 1\, d\theta \cdot \int_0^{\pi/4} \sin\varphi\, d\varphi && \text{(separable)} \\
&= \left[\frac{\rho^3}{3}\right]_0^1 \cdot [\theta]_0^{2\pi} \cdot [-\cos\varphi]_0^{\pi/4} \\
&= \left(\frac{1}{3}\right)(2\pi)\left(-\cos(\pi/4) + \cos(0)\right) \\
&= \frac{2\pi}{3}\left(-\frac{\sqrt{2}}{2} + 1\right) \\
&= \frac{2\pi}{3} \cdot \frac{2 - \sqrt{2}}{2} \\
&= \frac{2\pi(2 - \sqrt{2})}{6} \\
&\approx 0.6.
\end{aligned}$$

Remark: I know that the most difficult part of this material is the visualization and parametrization of regions in $\mathbb{R}^3$. I strongly encourage you to use visualization tools such as Desmos and GeoGebra to build intuition:

`https://www.desmos.com/3d`
`https://www.geogebra.org/3d`