Math 310 **Fall 2023**
Homework 4 **Drew Armstrong**

Problem 1. An Integral in the Plane. Consider the function $f(x,y) = x$. Let D be the region that is inside the circle $x^2 + y^2 = 4$, above the line $y = 0$ and below the line $y = x$.

(a) Draw the region. [Hint: It looks like 1/8 of a pie.]
(b) Compute the integral $\iint_D f(x,y)\,dxdy$ by converting to polar coordinates.
(c) Compute the integral $\iint_D f(x,y)\,dxdy$ in Cartesian coordinates by cutting the region D into two pieces D_1 and D_2 separated by the line $x = \sqrt{2}$. Check that you answers from parts (a) and (b) are the same.

Problem 2. Center of Mass. Let D be the same region as in Problem 1. Think of this as a thin metal plate with a constant density of 1 unit of mass per unit of area. Compute the following using polar coordinates.

(a) Compute the total mass $\iint_D 1\,dxdy$.
(b) Compute the moment about the y axis: $\iint_D x\,dxdy$.
(c) Compute the moment about the x axis: $\iint_D y\,dxdy$.
(d) Find the center of mass.

Problem 3. Change of Coordinates. Consider the function $f(x,y) = x^2 + y^2$. Let D be the square-shaped region in the x,y-plane bounded by the four lines $x + y = \pm 2$ and $x - y = \pm 2$.

(a) Draw the region.
(b) Consider the change of variables $x = u + v$ and $y = u - v$. Compute the area stretch factor (i.e., the absolute value of the determinant of the Jacobian matrix.)
(c) Compute the integral $\iint_D (x^2 + y^2)\,dxdy$ by converting to u,v-coordinates. [Hint: The region D in the u,v-plane is parametrized by $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$.]

Problem 4. Integration Over a Rectangular Box. Let B be the rectangular box parametrized by $0 \leq x \leq 1$, $0 \leq y \leq 2$ and $0 \leq z \leq 3$. Compute the triple integral

$$\iiint_B (x + y + z)\,dxdydz.$$

Problem 5. Cylindrical Coordinates. Consider a solid cone of radius 1 and height 1 whose base is the unit disk $x^2 + y^2 \leq 1$ in the x,y-plane and whose vertex is at the point $(0,0,1)$ in x,y,z-space.

(a) Parametrize the cone using cylindrical coordinates: r, θ, z.
(b) Compute the volume of the cone.
(c) Compute the center of mass $(\bar{x}, \bar{y}, \bar{z})$, assuming that the cone has constant density 1. [Hint: By symmetry we know that $\bar{x} = 0$ and $\bar{y} = 0$, so you only have to compute $\bar{z}$.]

Problem 6. Spherical Coordinates. Consider the "ice-cream-cone-shaped" solid region E that is between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z^2 = x^2 + y^2$, and satisfies $z \geq 0$. The volume is given by the triple integral:

$$\text{Vol}(E) = \iiint_E 1\,dxdydz.$$

Compute this integral by converting to spherical coordinates.