Math 310 **Fall 2023**
Homework 3 **Drew Armstrong**

Problem 1. Tangent Line to an Ellipse. Let $a, b > 0$ and consider the ellilpse

$$ax^2 + by^2 = 1.$$

(a) Let (x_0, y_0) be any point satisfying $ax_0^2 + by_0^2 = 1$. Show that the tangent line to the ellipse at the point (x_0, y_0) has the equation

$$ax_0x + by_0y = 1.$$

[Hint: Think of the ellipse as the level curve $f(x, y) = 1$ where $f(x, y) = ax^2 + by^2$.]

(b) Draw the ellipse and tangent line when $a = 1$, $b = 3$ and $(x_0, y_0) = (1/2, 1/2)$.

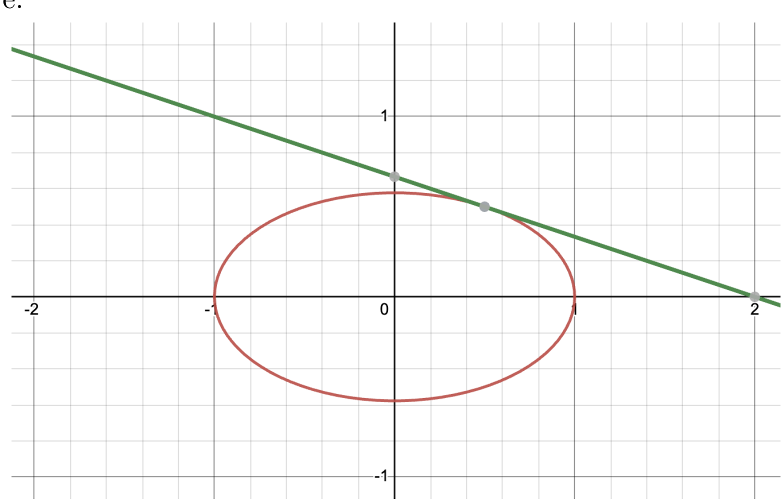
(a): Consider the function $f(x, y) = ax^2 + by^2$ with gradient vector field $\nabla f(x, y) = \langle 2ax, 2by\rangle$. Let (x_0, y_0) be any point on the level curve $f(x, y) = 1$, so that $ax_0^2 + by_0^2 = 1$. Then the equation of the tangent line to the level curve $f(x, y) = 1$ at the point (x_0, y_0) is

$$\begin{aligned}
\nabla f(x_0, y_0) \bullet \langle x - x_0, y - y_0\rangle &= 0 \\
\langle 2ax_0, 2by_0\rangle \bullet \langle x - x_0, y - y_0\rangle &= 0 \\
2ax_0(x - x_0) + 2by_0(y - y_0) &= 0 \\
2ax_0x - 2ax_0^2 + 2by_0y - 2by_0^2 &= 0 \\
2ax_0x + 2by_0y &= 2ax_0^2 + 2by_0^2 \\
ax_0x + by_0y &= ax_0^2 + by_0^2 \\
ax_0x + by_0y &= 1.
\end{aligned}$$

(b): We consider the case when $a = 1$, $b = 3$ and $(x_0, y_0) = (1/2, 1/2)$. From part (a) we know that the equation of the tangent line to the ellipse $1x^2 + 3y^2 = 1$ at the point $(1/2, 1/2)$ is

$$\begin{aligned}
ax_0x + by_0y &= 1 \\
1(1/2)x + 3(1/2)y &= 1 \\
3y/2 &= -x/2 + 1 \\
y &= -x/3 + 2/3.
\end{aligned}$$

Here is a picture:



Problem 2. Tangent Plane to a Surface. Consider the scalar field $f(x,y,z) = xye^z$.

(a) Compute the gradient vector field $\nabla f(x,y,z)$.

(b) Use your answer from part (a) to find the equation of the tangent plane to the level surface $f(x,y,z) = 2$ at the point $(x_0,y_0,z_0) = (2,1,0)$.

(a): The function $f(x,y,z) = xye^z$ has gradient vector field

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle ye^z, xe^z, xye^z \rangle .$$

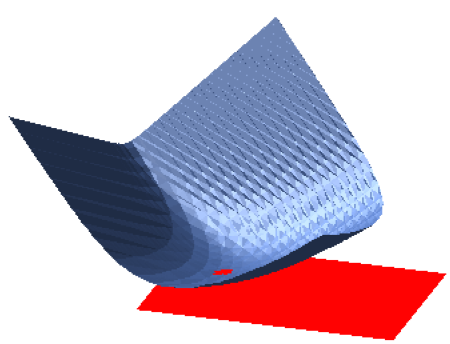
(b): If $f(x_0,y_0,z_0) = c$ then the tangent plane to the level curve $f(x,y,z) = c$ at the point (x_0,y_0,z_0) has the equation

$$\begin{aligned} \nabla f(x_0,y_0,z_0) \bullet \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ \frac{\partial f}{\partial x}(x_0,y_0,z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0,z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0,y_0,z_0)(z - z_0) &= 0 \\ y_0e^{z_0}(x - x_0) + x_0e^{z_0}(y - y_0) + x_0y_0e^{z_0} &= 0. \end{aligned}$$

Putting $(x_0,y_0,z_0) = (2,1,0)$ gives equation

$$\begin{aligned} 1e^0(x - 2) + 2e^0(y - 1) + 2 \cdot 1 \cdot e^0(z - 0) &= 0 \\ (x - 2) + 2(y - 1) + 2(z - 0) &= 0 \\ x - 2 + 2y - 2 + 2z &= 0 \\ x + 2y + 2z &= 4. \end{aligned}$$

GeoGebra doesn't like to plot implicit surfaces so I used a program called Maple to draw this:



Problem 3. Gradient Flow. The concentration of algae in a shallow pond is given by

$$A(x,y) = x^2 + 3y^2.$$

A certain fish always swims in the direction of **maximum increase of algae**. If $\mathbf{r}(t)$ is the position of the fish at time t, this means that the velocity $\mathbf{r}'(t)$ and the gradient vector $\nabla A(\mathbf{r}(t))$ must always be parallel.

(a) Show that the path $\mathbf{r}(t) = (e^{2t}, e^{6t})$ has this property.

(b) Show that the path $\mathbf{r}(t) = (t, t^3)$ also has this property.

(a): The gradient vector field is $\nabla A(x, y) = \langle 2x, 6y \rangle$. If the fish travels along the trajectory $\mathbf{r}(t) = (e^{2t}, e^{6t})$ then the fish's velocity vector at time t is $\mathbf{r}'(t) = \langle 2e^{2t}, 6e^{6t} \rangle$. On the other hand, the algae gradient at the fish's position is

$$\nabla A(\mathbf{r}(t)) = \nabla A(e^{2t}, e^{6t}) = \langle 2e^{2t}, 6e^{6t} \rangle.$$

We note that $\mathbf{r}'(t)$ and $\nabla A(\mathbf{r}(t))$ are equal, hence they are certainly parallel.

(b): This time we consider the trajectory $\mathbf{r}(t) = (t, t^3)$ with velocity $\mathbf{r}'(t) = \langle 1, 3t^2 \rangle$. The algae gradient at the fish's position is

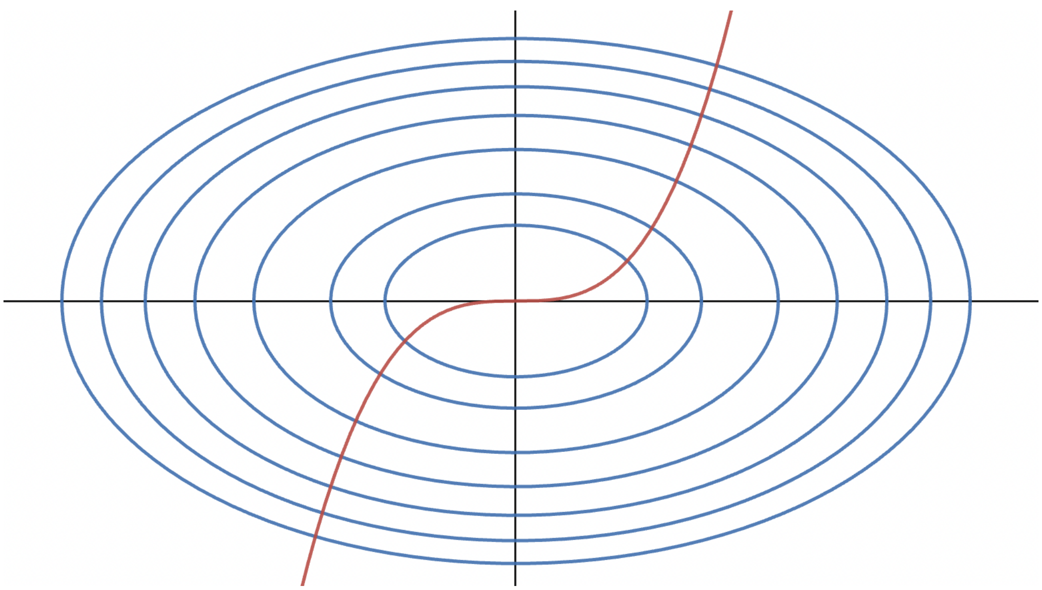
$$\nabla A(\mathbf{r}(t)) = \nabla A(t, t^3) = \langle 2t, 6t^3 \rangle.$$

This vector is not exactly equal to the fish's velocity, but it is still parallel because

$$\nabla A(\mathbf{r}(t)) = \langle 2t, 6t^3 \rangle = 2t\langle 1, 3t^2 \rangle = 2t\mathbf{r}'(t).$$

If $t > 0$ then $2t$ is a positive scalar so the vectors $\nabla A(\mathbf{r}(t))$ and $\mathbf{r}'(t)$ are always parallel.

Remark: Both of the trajectories $\langle x(t), y(t) \rangle = \langle e^{2t}, e^{6t} \rangle$ and $\langle x(t), y(t) \rangle = \langle t, t^3 \rangle$ satisfy $x(t)^3 = y(t)$ for all t, hence they travel within the curve $y = x^3$. This curve (in red) is perpendicular to every level curve of the function $A(x, y)$ (in blue):



Problem 4. Differentials. Let ℓ, w, h be the length, width and height of a box with an open top. The volume and surface area of the box are

$$V(\ell, w, h) = \ell wh,$$
$$A(\ell, w, h) = \ell w + 2\ell h + 2wh.$$

(a) Use the multivariable chain rule to express the differentials dV and dA in terms of the values of w, ℓ, h and the differentials $dw, d\ell, dh$.
(b) Suppose that you measure ℓ, w, h to be $10, 11, 12$ cm, respectively, each with a maximum error of 0.1 cm. Use your answer from (a) to find the **approximate error** in the computed values of V and A. [Hint: Substitute 0.1 for dw, $d\ell$ and dh.]

(a): The differential of the function $V(\ell, w, h) = \ell wh$ is

$$dV = \frac{\partial V}{\partial \ell}d\ell + \frac{\partial V}{\partial w}dw + \frac{\partial V}{\partial h}dh = whd\ell + \ell hdw + \ell wdh.$$

The differential of the function $A(\ell, w, h) = \ell w + 2\ell h + 2wh$ is

$$dA = \frac{\partial A}{\partial \ell}d\ell + \frac{\partial A}{\partial w}dw + \frac{\partial A}{\partial h}dh = (w + 2h)d\ell + (\ell + 2h)dw + (2\ell + 2w)dh.$$

(b): Suppose we measure $\ell = 10$, $w = 11$ and $h = 12$ cm with uncertainties $dw = d\ell = dh = 0.1$. Then the computed volume is $V = (10)(11)(12) = 1320 \text{ cm}^3$ with uncertainty

$$dV = (11)(12)(0.1) + (10)(12)(0.1) + (10)(11)(0.1) = 36.2 \text{ cm}^3$$

and the computed area is $A = (10)(11) + 2(10)(12) + 2(11)(12) = 614 \text{ cm}^2$ with uncertainty

$$dA = [(11) + 2(12)](0.1) + [(11) + 2(12)](0.1) + [2(10)(12) + 2(11)(12)](0.1) = 11.1 \text{ cm}^2.$$

Remark: We introduced the differential notation (dV) in this section because we will use it later when discussing multivariable integrals. The error estimates that we just computed are more correctly seen as linear approximations. For example, if ℓ_0, w_0, h_0 are the measured values of ℓ, w, h and $V_0 = \ell_0 w_0 h_0$ is the computed value of V, then linear approximation says

$$\begin{aligned} V - V_0 &\approx w_0h_0(\ell - \ell_0) + \ell_0h_0(w - w_0) + w_0\ell_0(h - h_0) \\ \Delta V &\approx w_0h_0\Delta\ell + \ell_0h_0\Delta w + w_0\ell_0\Delta h. \end{aligned}$$

It's just a slightly different notation. (There are too many notations.)

Problem 5. Multivariable Optimization. Consider the scalar field $f(x, y) = x^3 + xy - y^3$.

(a) Compute the gradient vector field $\nabla f(x, y)$.
(b) Find all the critical points of f, i.e., points (a, b) such that $\nabla f(a, b) = \langle 0, 0\rangle$.
(c) Compute the Hessian determinant $\det(Hf)$.
(d) Use the "second derivative test" to determine whether each critical point from part (b) is a local maximum, local minimum or a saddle point.

(a): The gradient vector field of the scalar field $f(x, y) = x^3 + xy - y^3$ is

$$\nabla f = \langle f_x, f_y\rangle = \langle 3x^2 + y, x - 3y^2\rangle.$$

(b): The critical points satisfy $\nabla f = \langle 0, 0\rangle$, which is a system of two nonlinear equations in two unknowns:

$$\begin{cases} 3x^2 + y &= 0, \\ x - 3y^2 &= 0. \end{cases}$$

Nonlinear systems cannot generally be solved by hand, but this one can because I chose it carefully. We can write the first equation as $y = -3x^2$ and then substitute this into the second equation:

$$\begin{aligned} x - 3y^2 &= 0 \\ x - 3(-3x^2)^2 &= 0 \\ x - 27x^4 &= 0 \\ x(1 - 27x^3) &= 0. \end{aligned}$$

This implies that $x = 0$ or $1 - 27x^3 = 0$, hence $x^3 = 1/27$. The number $1/27$ has a unique real cube root $1/3$. Hence we conclude that $x = 0$ or $x = 1/3$. Since $y = -3x^2$, we obtain two critical points:

$$(0, 0) \quad \text{and} \quad (1/3, -1/3).$$

(c): The Hessian matrix is

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 1 \\ 1 & -6y \end{pmatrix}.$$

The Hessian determinant is

$$\det(Hf) = (6x)(-6y) - (1)(1) = -36xy - 1.$$

(d): The critical point $(0, 0)$ has

$$\det(Hf)(0, 0) = -36(0)(0) - 1 = -1 < 0,$$

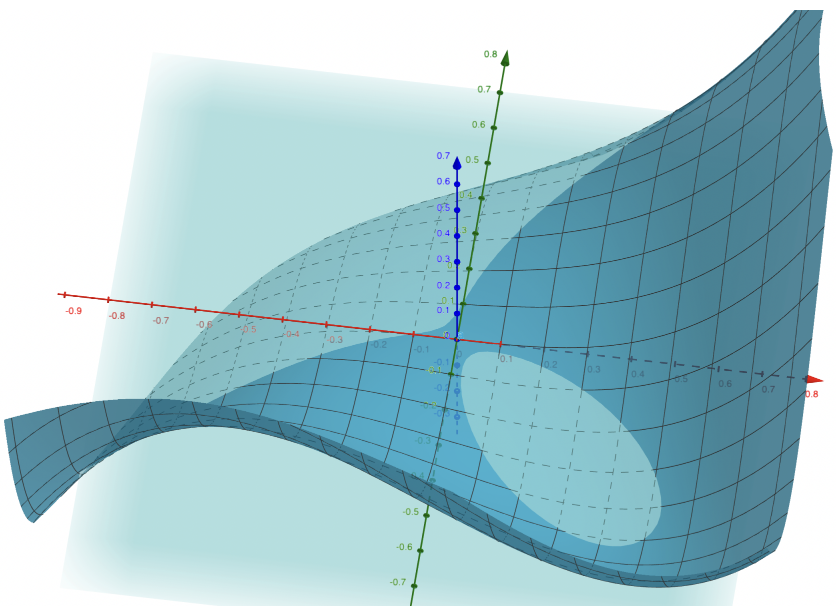
so it is a saddle. The critical point $(1/3, -1/3)$ has

$$\det(Hf)(1/3, -1/3) = -36(1/3)(-1/3) - 1 = 4 - 1 = 3 > 0,$$

so it is local maximum or minimum. To tell the difference we observe that

$$f_{xx}(1/3, -1/3) = 6(1/3) = 2 > 0,$$

so $(1/3, -1/3)$ is a local minimum. Here is a picture produced by GeoGebra:



Problem 6. Least Squares Regression. Suppose we have n points in the plane:

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots \quad (x_n, y_n).$$

We would like to find the line $y = mx + b$ that is "closest" to these points. The standard approach is to find values of m and b so the following "sum of squared errors" is minimized:

$$E(m, b) = (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \cdots + (y_n - mx_n - b)^2.$$

(a) Show that the equation $\partial E/\partial b = 0$ implies

$$m \sum x_i + nb = \sum y_i.$$

(b) Show that the equation $\partial E/\partial m = 0$ implies

$$m \sum x_i^2 + b \sum x_i = \sum x_i y_i.$$

(c) Solve these equations to find m and b when the given points are as follows:

$$(0,1), \quad (1,2), \quad (2,2), \quad (3,3).$$

Draw a picture of the points and the best fit line.

(a): We compute the partial derivative of E with respect to b:

$$\begin{aligned} E &= \sum (y_i - mx_i - b)^2 \\ \partial E/\partial b &= \tfrac{\partial}{\partial b} \sum (y_i - mx_i - b)^2 \\ &= \sum \tfrac{\partial}{\partial b} (y_i - mx_i - b)^2 \\ &= \sum 2(y_i - mx_i - b)(-1) \\ &= \sum (-2y_i + 2mx_i + 2b) \\ &= -2 \sum y_i + 2m \sum x_i + 2b \sum 1 \\ &= -2 \sum y_i + 2m \sum x_i + 2bn. \end{aligned}$$

(Note that the sum of 1 over $i = 1, \ldots, n$ is $1 + 1 + \cdots + 1 = n$.) Setting $\partial E/\partial b = 0$ gives

$$\begin{aligned} -2 \sum y_i + 2m \sum x_i + 2bn &= 0 \\ -\sum y_i + m \sum x_i + bn &= 0 \\ m \sum x_i + bn &= \sum y_i. \end{aligned}$$

(b): We compute the partial derivative of E with respect to m:

$$\begin{aligned} E &= \sum (y_i - mx_i - b)^2 \\ \partial E/\partial m &= \tfrac{\partial}{\partial m} \sum (y_i - mx_i - b)^2 \\ &= \sum \tfrac{\partial}{\partial m} (y_i - mx_i - b)^2 \\ &= \sum 2(y_i - mx_i - b)(-x_i) \\ &= \sum (-2x_i y_i + 2mx_i^2 + 2bx_i) \\ &= -2 \sum x_i y_i + 2m \sum x_i^2 + 2b \sum x_i. \end{aligned}$$

Setting $\partial E/\partial m = 0$ gives

$$\begin{aligned} -2 \sum x_i y_i + 2m \sum x_i^2 + 2b \sum x_i &= 0 \\ -\sum x_i y_i + m \sum x_i^2 + b \sum x_i &= 0 \\ m \sum x_i^2 + b \sum x_i &= \sum x_i y_i. \end{aligned}$$

(c): We want to find the line $y = mx + b$ that is "closest" to the $n = 4$ points

$$\begin{aligned} (x_1, y_1) &= (0, 1), \\ (x_2, y_2) &= (1, 2), \end{aligned}$$

$$(x_3, y_3) = (2, 2),$$
$$(x_4, y_4) = (3, 3).$$

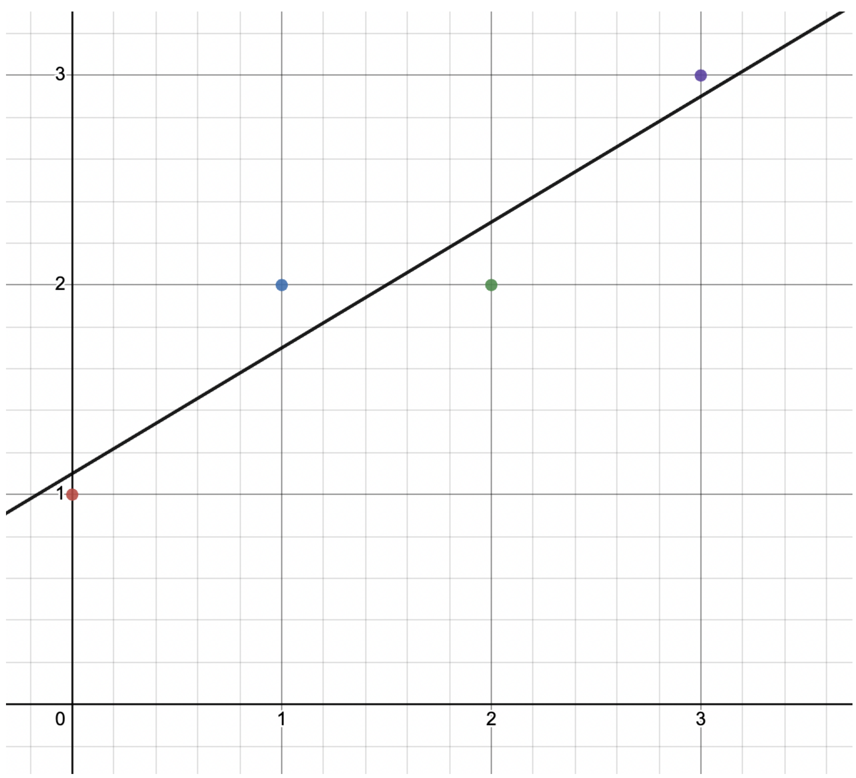
From these points we compute

$$\sum x_i = 0 + 1 + 2 + 3 = 6,$$
$$\sum x_i^2 = 0 + 1 + 4 + 9 = 14,$$
$$\sum x_i y_i = 0 + 2 + 4 + 9 = 15,$$
$$\sum y_i = 1 + 2 + 2 + 3 = 8.$$

Hence from parts (a) and (b) the unknown slope and y-intercept m, b satisfy the following system of two linear equations, which are called the "normal equations":

$$\begin{cases} 6m + 4b &= 8 \\ 14m + 6b &= 15. \end{cases}$$

The solution is $m = 3/5$ and $b = 11/10$ (I was tired so I used a computer), hence the best fit line is $y = (3/5)x + (11/10)$. Here is a picture:



I can tell that the calculations were correct because the picture looks good, i.e., the line looks like a "good fit" for the four data points.

Remark: With more work, one could check that $\det(HE)(3/5, 11/10) > 0$ and $E_{mm}(3/5, 11/10) > 0$, to verify that this really is a minimum. But of course it is. In most practical applications there is no need to use the second derivative test.