Math 310 **Fall 2023**
Homework 3 **Drew Armstrong**

Problem 1. Tangent Line to an Ellipse. Let $a, b > 0$ and consider the ellilpse

$$ax^2 + by^2 = 1.$$

(a) Let (x_0, y_0) be any point satisfying $ax_0^2 + by_0^2 = 1$. Show that the tangent line to the ellipse at the point (x_0, y_0) has the equation

$$ax_0x + by_0y = 1.$$

[Hint: Think of the ellipse as the level curve $f(x, y) = 1$ where $f(x, y) = ax^2 + by^2$.]

(b) Draw the ellipse and tangent line when $a = 1$, $b = 3$ and $(x_0, y_0) = (1/2, 1/2)$.

Problem 2. Tangent Plane to a Surface. Consider the scalar field $f(x, y, z) = xye^z$.

(a) Compute the gradient vector field $\nabla f(x, y, z)$.

(b) Use your answer from part (a) to find the equation of the tangent plane to the level surface $f(x, y, z) = 2$ at the point $(x_0, y_0, z_0) = (2, 1, 0)$.

Problem 3. Gradient Flow. The concentration of algae in a shallow pond is given by

$$A(x, y) = x^2 + 3y^2.$$

A certain fish always swims in the direction of **maximum increase of algae**. If $\mathbf{r}(t)$ is the position of the fish at time t, this means that the velocity $\mathbf{r}'(t)$ and the gradient vector $\nabla A(\mathbf{r}(t))$ must always be parallel.

(a) Show that the path $\mathbf{r}(t) = (e^{2t}, e^{6t})$ has this property.

(b) Show that the path $\mathbf{r}(t) = (t, t^3)$ also has this property.

Problem 4. Differentials. Let ℓ, w, h be the length, width and height of a box with an open top. The volume and surface area of the box are

$$V(\ell, w, h) = \ell wh,$$
$$A(\ell, w, h) = \ell w + 2\ell h + 2wh.$$

(a) Use the multivariable chain rule to express the differentials dV and dA in terms of the values of w, ℓ, h and the differentials $dw, d\ell, dh$.

(b) Suppose that you measure ℓ, w, h to be $10, 11, 12$ cm, respectively, each with a maximum error of 0.1 cm. Use your answer from (a) to find the **approximate error** in the computed values of V and A. [Hint: Substitute 0.1 for dw, $d\ell$ and dh.]

Problem 5. Multivariable Optimization. Consider the scalar field $f(x, y) = x^3 + xy - y^3$.

(a) Compute the gradient vector field $\nabla f(x, y)$.

(b) Find all the critical points of f, i.e., points (a, b) such that $\nabla f(a, b) = \langle 0, 0 \rangle$.

(c) Compute the Hessian determinant $\det(Hf)$.

(d) Use the "second derivative test" to determine whether each critical point from part (b) is a local maximum, local minimum or a saddle point.

Problem 6. Least Squares Regrssion. Suppose we have n points in the plane:

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots \quad (x_n, y_n).$$

We would like to find the line $y = mx + b$ that is "closest" to these points. The standard approach is to find values of m and b so the following "sum of squared errors" is minimized:

$$E(m, b) = (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \cdots + (y_n - mx_n - b)^2.$$

(a) Show that the equation $\partial E/\partial b = 0$ implies

$$m \sum x_i + nb = \sum y_i.$$

(b) Show that the equation $\partial E/\partial m = 0$ implies

$$m \sum x_i^2 + b \sum x_i = \sum x_i y_i.$$

(c) Solve these equations to find m and b when the given points are as follows:

$$(0, 1), \quad (1, 2), \quad (2, 2), \quad (3, 3).$$

Draw a picture of the points and the best fit line.