Math 310 — Fall 2023
Homework 2 — Drew Armstrong

Problem 1. A Line in Space. Consider the line in $\mathbb{R}^3$ passing through the two points

$$P = (-1, 3, 2) \quad \text{and} \quad Q = (2, 5, 1).$$

(a) Express this line in parametric form $\mathbf{r}(t) = (x_0 + ta, y_0 + tb, z_0 + tc)$.
(b) Find the equations of two planes in $\mathbb{R}^3$ whose intersection is this line. [Hint: There are infinitely many solutions. One solution uses the symmetric equations.]

Problem 2. A Plane in Space. Consider the plane in $\mathbb{R}^3$ passing through the three points

$$P = (-1, 3, 2), \quad Q = (2, 5, 1), \quad R = (0, 2, 4).$$

(a) Find a vector that is perpendicular to this plane.
(b) Find the equation of the plane.

Problem 3. Intersection of Two Planes. Consider the following two planes in $\mathbb{R}^3$:

$$\begin{cases} x + y + 2z = 1, \\ x - y + z = 3. \end{cases}$$

(a) Express the intersection of these planes as a parametrized line. [Hint: Subtract the equations to obtain a new equation without x. Then let $t = z$ be a parameter and solve for x and y in terms of t.]
(b) We observe that $\mathbf{u} = \langle 1, 1, 2 \rangle$ and $\mathbf{v} = \langle 1, -1, 1 \rangle$ are normal vectors for the two planes. Compute the cross product $\mathbf{u} \times \mathbf{v}$. How is this vector related to the line in part (a)?

Problem 4. Projectile Motion. A projectile is launched from the point $(0, 0)$ in $\mathbb{R}^2$ with an initial speed of s, at an angle of θ above the horizontal. Thus we have

$$\mathbf{r}(0) = \langle 0, 0 \rangle,$$
$$\mathbf{r}'(0) = \langle s \cos\theta, s \sin\theta \rangle.$$

Let $g > 0$ be the constant of gravity (which is 9.81 m/s^2 near the Earth).

(a) Use this information to compute the position $\mathbf{r}(t)$ at time t. [Hint: Neglecting air resistance, the acceleration due to gravity is constant: $\mathbf{r}''(t) = \langle 0, -g \rangle$.]
(b) Show that the particle travels a horizontal distance of $H = s^2 \sin(2\theta)/g$ before it hits the ground. [Hint: Use your answer $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ from part (a) and solve the equation $y(t) = 0$ for t. You will need the trig identity $\sin(2\theta) = 2 \sin\theta \cos\theta$.]
(c) Find the value of θ that maximizes the horizontal distance traveled. [Hint: According to Calculus I, you should find the value of θ that makes $dH/d\theta = 0$. Recall that g and s are constant.]

Problem 5. Fun with the Product Rule. Recall the following "product rules" for vector-valued functions $\mathbf{f}, \mathbf{g} : \mathbb{R} \to \mathbb{R}^3$:

$$[\mathbf{f}(t) \bullet \mathbf{g}(t)]' = \mathbf{f}'(t) \bullet \mathbf{g}(t) + \mathbf{f}(t) \bullet \mathbf{g}'(t),$$
$$[\mathbf{f}(t) \times \mathbf{g}(t)]' = \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t).$$

(a) Let $\mathbf{r}(t)$ be the trajectory of a particle traveling on the surface of a sphere centered at $(0, 0, 0)$. In this case, show that $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all t. [Hint: By assumption we have $\|\mathbf{r}(t)\| = c$ for some constant c independent of t.]

(b) Let $\mathbf{r}(t)$ be the trajectory of a particle in space, and assume that $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar function $c(t)$. In this case show that

$$[\mathbf{r}(t) \times \mathbf{r}'(t)]' = \langle 0, 0, 0 \rangle \quad \text{for all } t.$$

[Hint: Recall that $\mathbf{v} \times \mathbf{v} = \langle 0, 0, 0 \rangle$ for any vector $\mathbf{v}$.]

Problem 6. Universal Gravitation. Choose a coordinate system with the sun at the origin $(0, 0, 0)$ in $\mathbb{R}^3$. According to Newton, a planet at position $\mathbf{r}(t)$ feels a gravitational force $\mathbf{F}(t)$ pointed directly toward the sun. The magnitude of this force is

$$\|\mathbf{F}(t)\| = \frac{GMm}{\|\mathbf{r}(t)\|^2},$$

where M is the mass of the sun, m is the mass of the planet and G is a constant of gravitation. For simplicity, let's assume that $G = M = m = 1$.

(a) Show that $\mathbf{F}(t) = -GMm\mathbf{r}(t)/\|\mathbf{r}(t)\|^3$. It follows from Newton's Second Law that

$$\mathbf{r}''(t) = -\frac{GM}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t).$$

[Hint: Since $\mathbf{F}(t)$ points directly toward the sun we must have $\mathbf{F}(t) = -c(t)\mathbf{r}(t)$ for some positive scalar $c(t)$, and hence $\|\mathbf{F}(t)\| = c(t)\|\mathbf{r}(t)\|$. Solve for $c(t)$.]

(b) **Conservation of Angular Momentum.** Consider the *angular momentum* vector

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t).$$

Use part (a) and Problem 5(b) to show that $\mathbf{L}'(t) = \langle 0, 0, 0 \rangle$ for all t. It follows that the angular momentum vector is constant.