Problem 1. Lines and Circles. The parametrized curve in part (a) is a line. The parametrized curve in part (b) is a circle. In each case, compute the velocity and speed at time t. Also eliminate t to find an equation for the curve in terms of x and y.

(a) $(x, y) = (p + ut, q + vt)$ where p, q, u, v are constants.
(b) $(x, y) = (a + r\cos(\omega t), b + r\sin(\omega t))$ where a, b, r, ω are constants.

[Oops: The solution uses the letters a and b instead of p and q.]

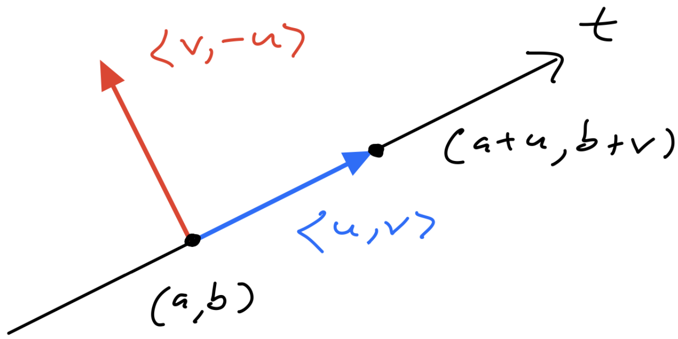
(a) **Line.** The velocity and speed are

$$(dx/dt, dy/dt) = (u, v) \quad \text{and} \quad \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{u^2 + v^2}.$$

Note that these are both constant, i.e., they do not depend on t. To eliminate t we will assume that $u \neq 0$ and $v \neq 0$, so that $x = a + ut$ implies $t = (x - a)/u$ and $y = b + vt$ implies $t = (y - b)/v$. Then equation these expressions for t gives

$$\begin{aligned} (x-a)/u &= (y-b)/v \\ v(x-a) &= u(y-b) \\ v(x-a) - u(y-b) &= 0. \end{aligned}$$

From our discussion in class we see that this line contains the point (a, b) and is perpendicular to the vector $\langle v, -u\rangle$. Here is a picture:



(b) **Circle.** The velocity and speed are

$$(dx/dt, dy/dt) = (-r\omega\sin(\omega t), r\omega\cos(\omega t))$$

and

$$\begin{aligned} \sqrt{(dx/dt)^2 + (dy/dt)^2} &= \sqrt{[-r\omega\sin(\omega t)]^2 + [r\omega\cos(\omega t)]^2} \\ &= \sqrt{r^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} \\ &= \sqrt{r^2\omega^2} \\ &= r\omega. \end{aligned}$$

We assume that r and ω are positive, so $\sqrt{r^2\omega^2} = |r\omega| = r\omega$. The speed is constant, but the velocity vector is not constant.[1] We can eliminate t by using the trig identity $\sin^2(\omega t) + \cos^2(\omega t) = 1$ as follows:

$$
\begin{aligned}
(x-a)^2 + (y-b)^2 &= [r\cos(\omega t)]^2 + [r\sin(\omega t)]^2 \\
(x-a)^2 + (y-b)^2 &= r^2[\cos^2(\omega t) + \sin^2(\omega t)] \\
(x-a)^2 + (y-b)^2 &= r^2.
\end{aligned}
$$

This is the equation of a circle with radius r, centered at (a, b).

Problem 2. Semi-Cubical Parabola. Consider the parametrized curve

$$(x, y) = (t^2, t^3).$$

(a) Eliminate t to find an equation relating x and y. [Hint: Note that $y/x = t$.]
(b) Compute the velocity and speed at time t.
(c) Find the slope of the tangent line at time t.
(d) Use the information in (b) and (c) to sketch the curve for t from $-\infty$ to ∞.

(a): Substitute $t = y/x$ into the equation $x = t^2$ to get

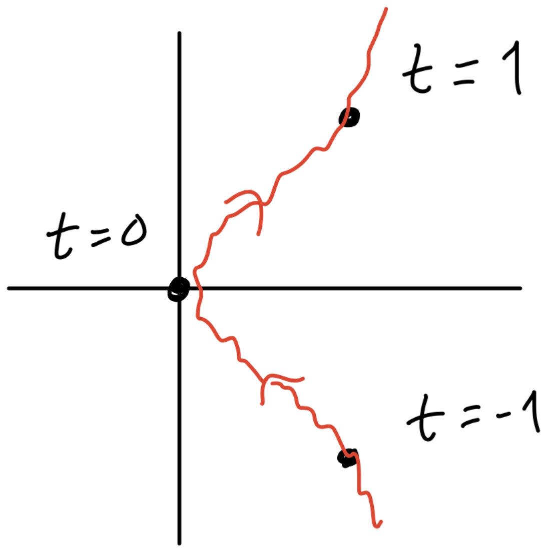
$$
\begin{aligned}
x &= (y/x)^2 \\
x &= y^2/x^2 \\
x^3 &= y^2.
\end{aligned}
$$

(c): Let's write $f(t) = (t^2, t^3)$. The velocity is $f'(t) = (dx/dt, dy/dt) = (2t, 3t^2)$, so the slope of the tangent line at time t is

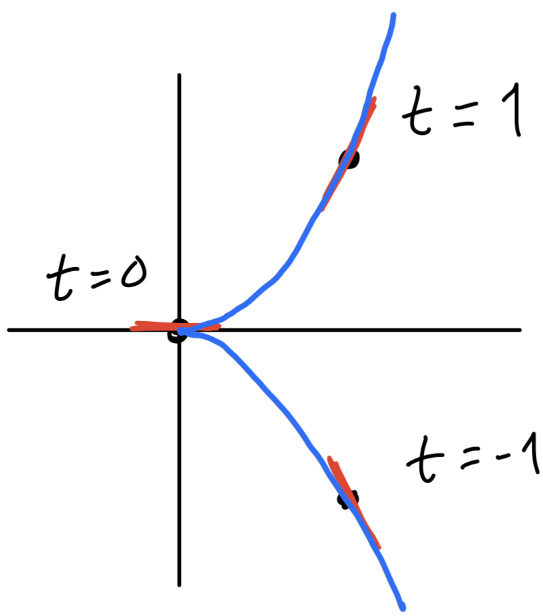
$$\frac{dy}{dx} = \frac{dx/dt}{dy/dt} = \frac{3t^2}{2t} = \frac{3}{2}t.$$

(c): In order to sketch the curve it is useful to note that $f(0) = (0, 0)$, $f(1) = (1, 1)$ and $f(-1) = (1, -1)$. Then the curve travels from $(1, -1)$ to $(0, 0)$ and then $(1, 1)$. But how does it travel?

[1] There is different vector, called the *angular velocity*, that is constant. It points out of the page into the third dimension and it has length $r\omega$.



In order to get more information we use the slope formula to sketch the tangent line at each point. The key fact is that the tangent is horizontal when $t = 0$. Thus the curve has a sharp "cusp". Here is a sketch:



[Remark: The point $f(0)$ is bad because $f'(0) = \langle 0, 0\rangle$ is the zero vector. Later we will say that this is a *critical point* of the path.]

Problem 3. The Cycloid. The cycloid is an interesting curve whose arc length can be computed by hand. It is parametrized by

$$(x, y) = (t - \sin t, 1 - \cos t).$$

(a) Check that the slope of the tangent at time t is $\sin t/(1-\cos t)$. Use this information to sketch the curve between $t = 0$ and $t = 2\pi$. [Hint: The slope goes to infinity when $t \to 0$ from the right and when $t \to 2\pi$ from the left. You do not need to prove this.]

(b) Compute the arc length between $t = 0$ and $t = 2\pi$. [Hint: You will need the trig identities $\sin^2 t + \cos^2 t = 1$ and $1 - \cos t = 2\sin^2(t/2)$.]

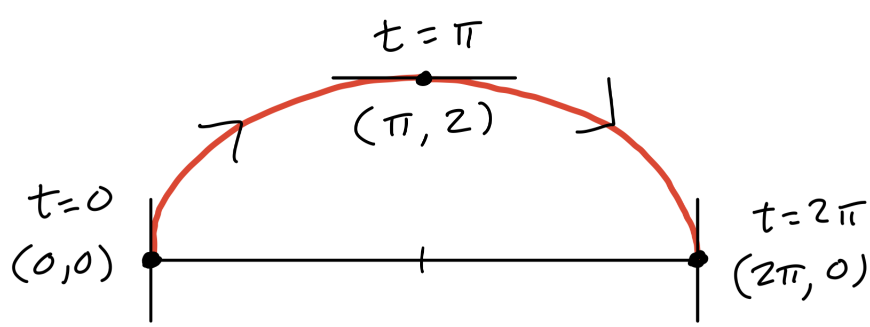
(a): Let $f(t) = (t - \sin t, 1 - \cos t)$, so the velocity is $f'(t) = (dx/dt, dy/dt) = (1 - \cos t, \sin t)$ and the slope of the tangent at time t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1-\cos t}.$$

The curve starts at the point $f(0) = (0,0)$, where the tangent is vertical because $\sin t/(1 - \cos t) \to +\infty$ as $t \to 0$ (from the right). The curve ends at $f(2\pi) = (2\pi, 0)$, where the tangent is again vertical because $\sin t/(1 - \cos t) \to -\infty$ as $t \to 2\pi$ (from the left).[2] Next we look for some time $0 < t < 2\pi$ when the slope of the tangent is zero:

$$\frac{\sin t}{1-\cos t} = 0 \quad \Rightarrow \quad \sin t = 0 \quad \Rightarrow \quad t = \pi.$$

Thus the curve has a horizontal tangent at the point $f(\pi) = (\pi, 2)$. Here is a picture:



(b): We can use the trig identities $\sin^2 t + \cos^2 t = 1$ and $1 - \cos t = 2\sin^2(t/s)$ to simplify the speed of the parametrization as follows:

$$\begin{aligned}\sqrt{(dx/dt)^2 + (dy/dt)^2} &= \sqrt{(1-\cos t)^2 + (\sin t)^2} \\ &= \sqrt{1 - 2\cos t + \cos^2 + \sin^2 t} \\ &= \sqrt{1 - 2\cos t + 1} \\ &= \sqrt{2 - 2\cos t} \\ &= \sqrt{2(1-\cos t)} \\ &= \sqrt{2 \cdot 2\sin^2(t/2)} \\ &= 2\sin(t/2),\end{aligned}$$

[2] You do not need to prove this. The limits can be computed with L'Hopital's rule.

which is non-negative because $0 \leq t \leq 2\pi$. Then the arc length between $t = 0$ and $t = 2\pi$ is the integral of the speed;

$$\begin{aligned}
\int_{t=0}^{t=2\pi} 2\sin(t/2)\,dt &= \int_{u=0}^{u=\pi} 2\sin u \cdot 2du \qquad\qquad [u = t/2, dt = 2du] \\
&= 4 \cdot [-\cos u]_{u=0}^{u=\pi} \\
&= 4 \cdot [-(-1) - (-1)] \\
&= 8.
\end{aligned}$$

Remarks:

- It is possible to eliminate t as follows. First we rewrite $y = 1 - \cos t$ as

$$\begin{aligned}
\cos t &= 1 - y \\
\cos^2 t &= 1 - 2y + y^2 \\
1 - \cos^2 t &= 2y - y^2 \\
\sin^2 t &= y(2 - y) \\
\sin t &= \sqrt{y(2-y)} \\
t &= \sin^{-1}\left(\sqrt{y(2-y)}\right).
\end{aligned}$$

  Then we substitute these expressions for t and $\sin t$ into the expression for x to get

$$x = t - \sin t = \sin^{-1}\left(\sqrt{y(2-y)}\right) - \sqrt{y(2-y)}.$$

  What a mess! Clearly it is better to express the cycloid in terms of a parametrization.

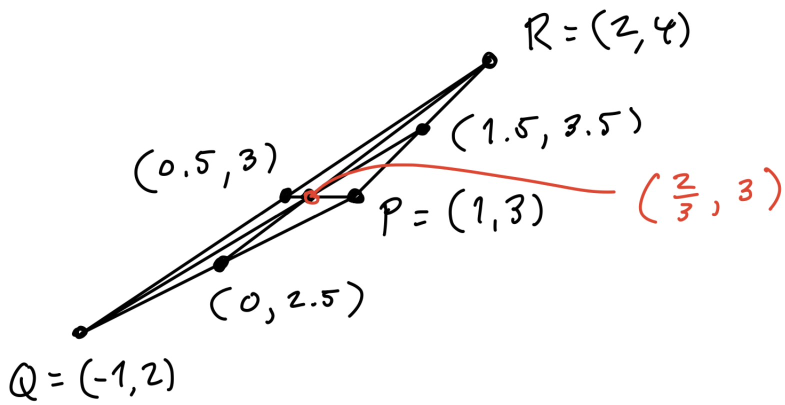
- The cycloid is the answer to several interesting problems in physics. For example, suppose you have a pebble stuck in the surface of your car tire. As the car moves the pebble will follow a cycloidal path. Suppose that the tire has radius 1 unit, so the circumference is 2π units. As your car travels a straight line distance of 2π units, the pebble will travel an arc length of 8 units.

Problem 4. A Triangle in the Plane. Consider the following points in $\mathbb{R}^2$:

$$P = (1,3), \quad Q = (-1,2), \quad R = (2,4).$$

(a) Draw the three points together with the midpoints $(P+Q)/2$, $(P+R)/2$, $(Q+R)/2$ and the center of mass $(P+Q+R)/3$.
(b) Find the coordinates of the three side vectors $\mathbf{u} = \vec{PQ}, \mathbf{v} = \vec{QR}, \mathbf{w} = \vec{PR}$.
(c) Use the length formula to compute the three side lengths $\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|$.
(d) Use the dot product to compute the three angles of the triangle.

(a): Oops, the points I gave you are rather cramped:



(b): Using the formula "head minus tail" gives

$$\begin{aligned}\mathbf{u} &= \langle(-1)-1, 2-3\rangle = \langle -2, -1\rangle,\\ \mathbf{v} &= \langle 2-(-1), 4-2\rangle = \langle 3, 2\rangle,\\ \mathbf{w} &= \langle 2-1, 4-3\rangle = \langle 1, 1\rangle.\end{aligned}$$

(c): The Pythagorean theorem gives

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{(-2)^2+(-1)^2} = \sqrt{5},\\ \|\mathbf{v}\| &= \sqrt{(-3)^2+(-2)^2} = \sqrt{13},\\ \|\mathbf{w}\| &= \sqrt{1^2+1^2} = \sqrt{2}.\end{aligned}$$

(d): Let α be the angle at P, which is the angle between vectors $\mathbf{u}$ and $\mathbf{w}$, so that

$$\cos\alpha = \frac{\mathbf{u}\bullet\mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|} = \frac{(-2)(1)+(-1)(1)}{\sqrt{5}\sqrt{2}} = \frac{-3}{\sqrt{10}} \quad\Rightarrow\quad \alpha = 161.56^\circ.$$

Let β be the angle at Q, which is the angle between vectors $-\mathbf{u}$ and $\mathbf{v}$, so that

$$\cos\beta = \frac{(-\mathbf{u})\bullet\mathbf{v}}{\|-\mathbf{u}\|\|\mathbf{v}\|} = \frac{(2)(3)+(1)(2)}{\sqrt{5}\sqrt{13}} = \frac{8}{\sqrt{65}} \quad\Rightarrow\quad \beta = 7.13^\circ.$$

Let γ be the angle at R, which is the angle between vectors $-\mathbf{v}$ and $-\mathbf{w}$, so that

$$\cos\gamma = \frac{(-\mathbf{v})\bullet(-\mathbf{w})}{\|-\mathbf{v}\|\|-\mathbf{w}\|} = \frac{(-3)(-1)+(-2)(-1)}{\sqrt{13}\sqrt{2}} = \frac{5}{\sqrt{26}} \quad\Rightarrow\quad \gamma = 11.31^\circ.$$

Check: These three angles do indeed add up to 180°.

[Remark: Note that $\theta > 90^\circ$ when $\cos\theta < 0$ and $0 < \theta < 90^\circ$ when $\cos\theta > 0$.]

Problem 5. A Triangle in Space. Consider the following points in $\mathbb{R}^3$:

$$P = (1,0,0), \quad Q = (1,1,0), \quad R = (1,1,1).$$

(a) Find the coordinates of the three side vectors $\mathbf{u} = \vec{PQ}, \mathbf{v} = \vec{QR}, \mathbf{w} = \vec{PR}$.
(b) Use the length formula to compute the three side lengths $\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|$.
(c) Use the dot product to compute the three angles of the triangle.

(a): Using the formula "head minus tail" gives

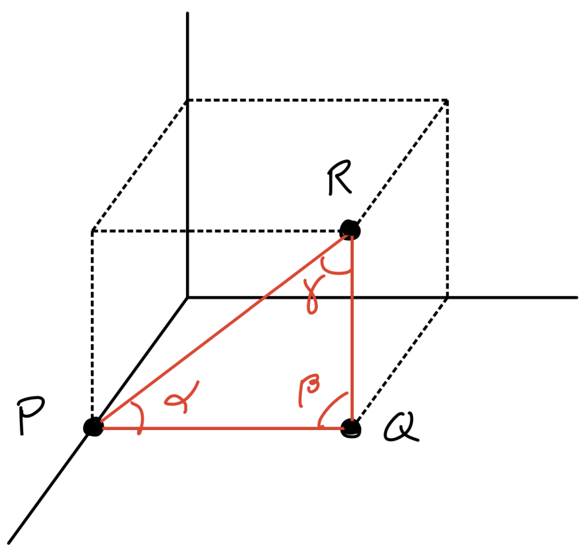
$$\mathbf{u} = \vec{PQ} = \langle 1-1, 1-0, 0-0\rangle = \langle 0,1,0\rangle,$$
$$\mathbf{v} = \vec{QR} = \langle 1-1, 1-1, 1-0\rangle = \langle 0,0,1\rangle,$$
$$\mathbf{w} = \vec{PR} = \langle 1-1, 1-0, 1-0\rangle = \langle 0,1,1\rangle.$$

(b): Using the formula for length gives

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\bullet\mathbf{u}} = \sqrt{0^2+1^2+0^2} = 1,$$
$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\bullet\mathbf{v}} = \sqrt{0^2+0^2+1^2} = 1,$$
$$\|\mathbf{w}\| = \sqrt{\mathbf{w}\bullet\mathbf{w}} = \sqrt{0^2+1^2+1^2} = \sqrt{2}.$$

We see from the side lengths that this is an isoceles right angled triangle, with angles 90°, 45°, 45°, but we will check it anyway.

(c): Consider the picture



First we compute the dot products:

$$\mathbf{u}\bullet\mathbf{v} = (0)(0)+(1)(0)+(0)(1) = 0,$$
$$\mathbf{u}\bullet\mathbf{w} = (0)(0)+(1)(1)+(0)(1) = 1,$$
$$\mathbf{v}\bullet\mathbf{w} = (0)(0)+(0)(1)+(1)(1) = 1.$$

Since α is the angle between $\mathbf{u}$ and $\mathbf{w}$ we have

$$\cos\alpha = \frac{\mathbf{u}\bullet\mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|} = \frac{1}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} \quad\Rightarrow\quad \alpha = 45^\circ.$$

Since β is the angle between $-\mathbf{u}$ and $\mathbf{v}$ we have

$$\cos\beta = \frac{(-\mathbf{u})\bullet\mathbf{v}}{\|-\mathbf{u}\|\|\mathbf{v}\|} = \frac{-(\mathbf{u}\bullet\mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{0}{1\cdot 1} = 0 \quad\Rightarrow\quad \beta = 90^\circ.$$

Since γ is the angle between $-\mathbf{v}$ and $-\mathbf{w}$ we have

$$\cos\gamma = \frac{(-\mathbf{v})\bullet(-\mathbf{w})}{\|-\mathbf{v}\|\|-\mathbf{w}\|} = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{1}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} \quad\Rightarrow\quad \gamma = 45^\circ.$$

Problem 6. Some Vector Arithmetic. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in 100-dimensional space. Use the properties of the dot product to show that

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}\bullet\mathbf{v}).$$

[Hint: Start with the definition $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v})$, then expand using FOIL.]

For any four vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ we can use the distributive rule for the dot product to get

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\bullet(\mathbf{c}+\mathbf{d}) &= \mathbf{a}\bullet(\mathbf{c}+\mathbf{d}) + \mathbf{b}\bullet(\mathbf{c}+\mathbf{d})\\ &= \mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{d}+\mathbf{b}\bullet\mathbf{c}+\mathbf{b}\bullet\mathbf{d}.\end{aligned}$$

This is a dot product version of FOIL. In our particular case we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\bullet\mathbf{u}-\mathbf{u}\bullet\mathbf{v}-\mathbf{v}\bullet\mathbf{u}+\mathbf{v}\bullet\mathbf{v}.\end{aligned}$$

Now we use the facts $\mathbf{u}\bullet\mathbf{u}=\|\mathbf{u}\|^2$, $\mathbf{v}\bullet\mathbf{v}=\|\mathbf{v}\|^2$ and $\mathbf{u}\bullet\mathbf{v}=\mathbf{v}\bullet\mathbf{u}$ to get

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= \mathbf{u}\bullet\mathbf{u}-\mathbf{u}\bullet\mathbf{v}-\mathbf{v}\bullet\mathbf{u}+\mathbf{v}\bullet\mathbf{v}\\ &= \mathbf{u}\bullet\mathbf{u}+\mathbf{v}\bullet\mathbf{v}-2(\mathbf{u}\bullet\mathbf{v})\\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2(\mathbf{u}\bullet\mathbf{v}).\end{aligned}$$

We discussed in class how this algebraic identity, together with the geometric Law of Cosines, leads to the theorem of the dot product:

$$\mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.$$

[Remark: In this solution I just assumed the basic properties of the dot product. For example, I assumed the distributive property:

$$\mathbf{a}\bullet(\mathbf{b}+\mathbf{c}) = \mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}.$$

Once upon a time someone had to prove this. Here is a proof: Let

$$\begin{aligned}\mathbf{a} &= \langle a_1,\ldots,a_n\rangle,\\ \mathbf{b} &= \langle b_1,\ldots,b_n\rangle,\\ \mathbf{c} &= \langle c_1,\ldots,c_n\rangle.\end{aligned}$$

Then we have

$$\begin{aligned}\mathbf{a}\bullet(\mathbf{b}+\mathbf{c}) &= \langle a_1,\ldots,a_n\rangle\bullet\langle b_1+c_1,\ldots,b_n+c_n\rangle\\ &= a_1(b_1+c_1)+\cdots+a_n(b_n+c_n)\\ &= a_1b_1+a_1c_1+\cdots+a_nb_n+a_nc_n\\ &= (a_1b_1+\cdots+a_nb_n)+\cdots+(a_1c_1+\cdots+a_nc_n)\\ &= \mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}.\end{aligned}$$

This is not a "proof-based class" so I didn't expect you to check this.]