Math 310 **Exam 1**
Fall 2023 **Wed Oct 4**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

Problem 1. Consider three points in space

$$P = (1, 1, 2), \quad Q = (3, 1, 1), \quad R = (2, 3, 4),$$

and consider the vectors $\mathbf{u} = Q - P = \langle 2, 0, -1\rangle$, $\mathbf{v} = R - P = \langle 1, 2, 2\rangle$.

(a) Compute the cross product $\mathbf{u} \times \mathbf{v}$ and use this to find the equation of the plane that passes through P, Q and R.

The cross product is

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \text{“}\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}\text{”} \\
&= \left\langle \det \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}, -\det \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \det \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \right\rangle \\
&= \langle 2, -5, 4\rangle.
\end{aligned}$$

Since the plane has normal vector $\mathbf{u} \times \mathbf{v} = \langle 2, -5, 4\rangle$ and passes through the point $P = (1, 1, 2)$ (for example), the equation of the plane is

$$\begin{aligned}
\langle 2, -5, 4\rangle \bullet \langle x - 1, y - 1, z - 2\rangle &= 0 \\
2(x - 1) - 5(y - 1) + 4(z - 2) &= 0 \\
2x - 5y + 4z - 2 + 5 - 8 &= 0 \\
2x - 5y + 4z &= 5.
\end{aligned}$$

(b) Compute the **area** of the triangle PQR. [Hint: This triangle is half of the parallelogram generated by $\mathbf{u}$ and $\mathbf{v}$.]

The area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} \times \mathbf{v}\| = \|\langle 2, -5, 4\rangle\| = \sqrt{2^2 + (-5)^2 + 4^2} = \sqrt{45}.$$

Hence the area of triangle PQR is $\sqrt{45}/2$.

Alternatively, let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Using the dot product gives

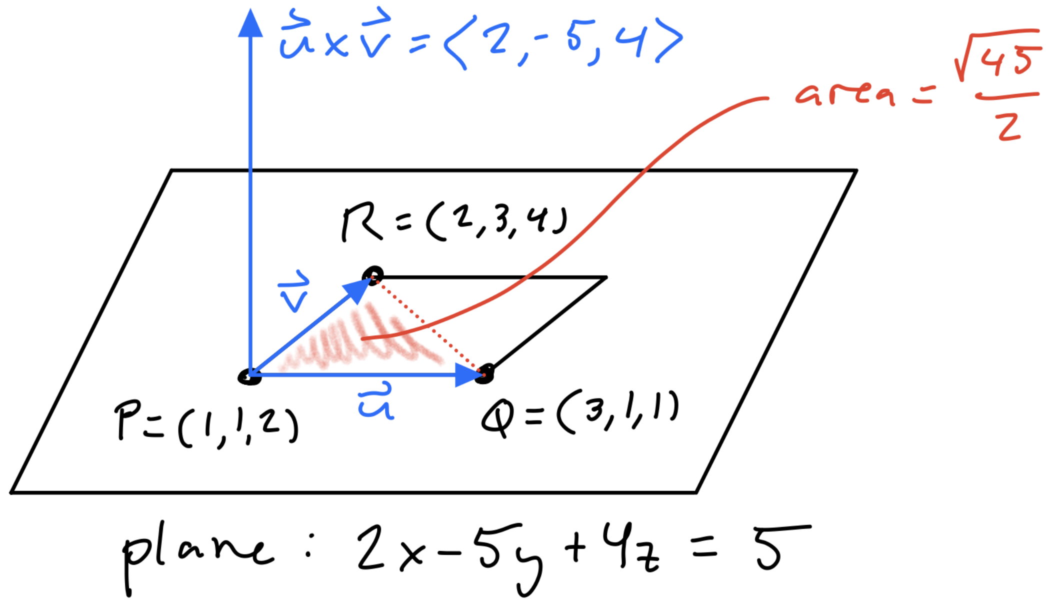
$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\sqrt{\mathbf{u} \bullet \mathbf{u}}\sqrt{\mathbf{v} \bullet \mathbf{v}}} = \frac{0}{\sqrt{5}\sqrt{9}} = 0.$$

(It's a right triangle.) Then the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \sqrt{5}\sqrt{9}\sqrt{1 - \cos^2\theta} = \sqrt{5}\sqrt{9}\sqrt{1 - 0} = \sqrt{45}.$$

Here is a picture of Problem 1:[1]

[1] You did not need to draw a picture.



Problem 2. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be the trajectory of a particle in the x, y-plane. Suppose there is a constant downward acceleration: $\mathbf{r}''(t) = \langle 0, -6\rangle$.

(a) If the initial velocity is $\mathbf{r}'(0) = \langle 1, 2\rangle$ and the initial position is $\mathbf{r}(0) = \langle 0, 0\rangle$, find the position at time t.

Integrate once to get the velocity:

$$\begin{aligned} \mathbf{r}'(t) &= \int \mathbf{r}''(t)\,dt \\ &= \langle \int 0\,dt, \int -6\,dt\rangle \\ &= \langle c_1, -6t + c_2\rangle. \end{aligned}$$

Substituting $t = 0$ gives $c_1 = 1$ and $c_2 = 2$, hence

$$\mathbf{r}'(t) = \langle 1, 2 - 6t\rangle.$$

Integrate again to get the position:

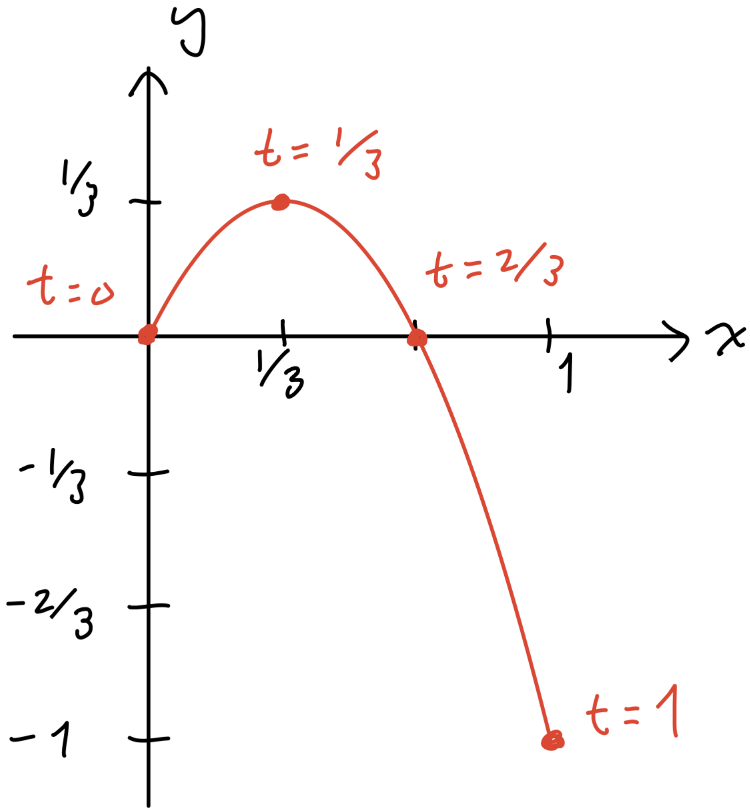
$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt \\ &= \langle \int 1\,dt, \int (2 - 6t)\,dt\rangle \\ &= \langle t + c_3, 2t - 3t^2 + c_4\rangle. \end{aligned}$$

Substituting $t = 0$ gives $c_3 = 0$ and $c_4 = 0$, hence

$$\mathbf{r}(t) = \langle t, -3t^2 + 2t\rangle.$$

Here is a picture of the trajectory between $t = 0$ and $t = 1$:[2]

[2]You did not need to draw a picture.



(b) Set up an integral to calculate the **arc length** traveled by the particle between $t = 0$ and $t = 1$. [This integral is too difficult to evaluate by hand.]

The arc length traveled between $t = 0$ and $t = 1$ is

$$\begin{aligned}\text{arc length} &= \int_0^1 \text{speed}\, dt \\ &= \int_0^1 \|\mathbf{r}'(t)\|\, dt \\ &= \int_0^1 \sqrt{1^2 + (2-6t)^2}\, dt\end{aligned}$$

(≈ 2.042 according to my computer).

Problem 3. A cylindrical can with radius r and height h has surface area

$$A(r,h) = 2\pi r^2 + 2\pi rh = 2\pi r(r+h).$$

(a) Use the chain rule to compute the differential dA in terms of dr and dh.

The chain rule tells us that

$$\begin{aligned}dA &= (\partial A/\partial r)dr + (\partial A/\partial h)dh \\ &= (4\pi r + 2\pi h)dr + 2\pi r dh.\end{aligned}$$

(b) Suppose you measure the can with a ruler to find that $r = 5$ cm and $h = 10$ cm, hence $A = 150\pi$ cm^2. If the sensitivity of the ruler is 0.1 cm, **estimate the error** in your computed value of A.

The result in part (a) is exactly true for infinitesimal differentials; it is approximately true for small finite differences:

$$\Delta A \approx (4\pi r + 2\pi h)\Delta r + 2\pi r \Delta h.$$

Substituting $r = 5$, $h = 10$, $\Delta r = 0.1$ and $\Delta h = 0.1$ gives

$$\Delta A \approx (20\pi + 20\pi)(0.1) + 10\pi(0.1) = 50\pi(0.1) = 5\pi.$$

[Remark: The percent error in the computed value of A is $5\pi/150\pi = 1/30 = 3.33\%$. The percent error in the measurements of r and h was 2% and 1%, respectively. I did not ask for the percent error.]

Problem 4. Consider the scalar field $f(x, y, z) = x^2yz$.

(a) Compute the gradient vector field ∇f.

The gradient vector field is

$$\begin{aligned}\nabla f &= \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z\rangle \\ &= \langle 2xyz, x^2z, x^2y\rangle.\end{aligned}$$

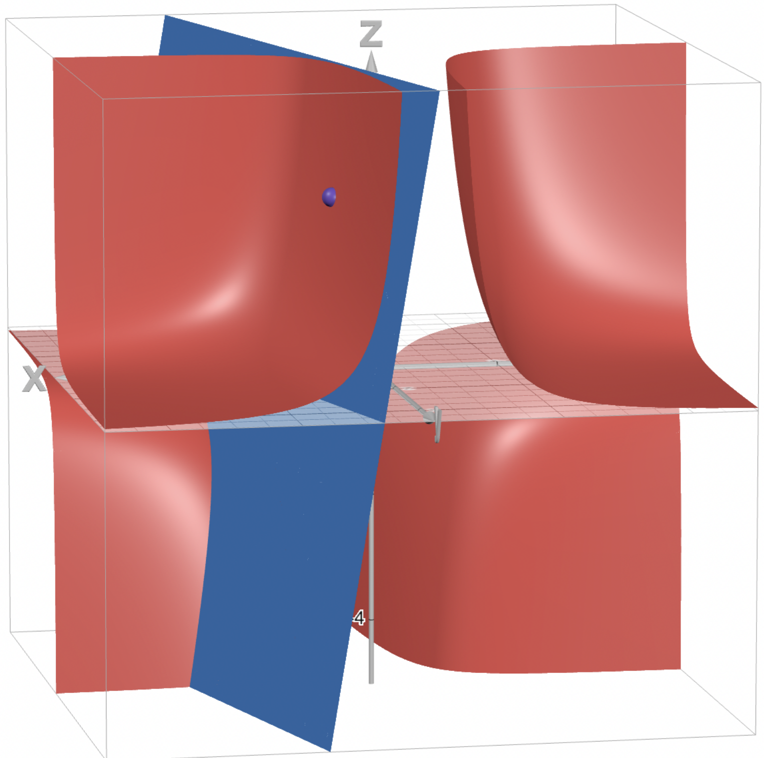
(b) Note that $f(1, 2, 3) = 1^2 \cdot 2 \cdot 3 = 6$. Use part (a) to find the **equation of the tangent plane** to the surface $x^2yz = 6$ at the point $(1, 2, 3)$.

The gradient vector $\nabla f(1, 2, 3) = \langle 1^2 \cdot 2 \cdot 3, 1^2 \cdot 3, 1^2 \cdot 2\rangle = \langle 12, 3, 2\rangle$ to the tangent plane at $(1, 2, 3)$. Hence the equation of the tangent plane is

$$\begin{aligned}\nabla f(1, 2, 3) \bullet \langle x - 1, y - 2, z - 3\rangle &= 0 \\ \langle 12, 3, 2\rangle \bullet \langle x - 1, y - 2, z - 3\rangle &= 0 \\ 12(x - 1) + 3(y - 2) + 2(z - 3) &= 0 \\ 12x + 3y + 2z - 12 - 6 - 6 &= 0 \\ 12x + 3y + 2z &= 24.\end{aligned}$$

Desmos has recently introduced 3D graphing[3] and it works really well. Here is a picture of the surface $x^2yz = 6$ and the tangent plane at $(1, 2, 3)$:

[3]It is currently in Beta. Type `desmos.com/3d` to get there.



Problem 5. The function $f(x, y) = -2x^3 + 3xy - y^3/4$ has critical points $(0, 0)$ and $(1, 2)$.

(a) Compute the Hessian matrix and its determinant.

We compute the first and second partial derivatives:

$$\begin{aligned} f_x &= -6x^2 + 3y, \\ f_y &= 3x - 3y^2/4, \\ f_{xx} &= -12x, \\ f_{xy} &= 3, \\ f_{yx} &= 3, \\ f_{yy} &= -3y/2. \end{aligned}$$

Hence the Hessian matrix is

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -12x & 3 \\ 3 & -3y/2 \end{pmatrix},$$

and its determinant is

$$\det(Hf) = (-12x)(-3y/2) - (3)(3) = 18xy - 9.$$

(b) Use the second derivative test to determine whether each of the two critical points is a local max, local min or a saddle point.

The critical point $(0, 0)$ has $\det(Hf)(0, 0) = -9 < 0$, hence it is a **saddle point**.

The critical point $(1, 2)$ has $\det(Hf)(1, 2) = 36 - 9 > 0$, hence is either a local max or min. To determine which, we can examine the sign of $f_{xx}(1, 2)$ or $f_{yy}(1, 2)$ (they have the same sign). Since $f_{xx}(1, 2) = -12(1) < 0$ we conclude that $(1, 2)$ is a **local maximum**. Here is a picture produced by Desmos: