Welcome to MTH 309

Course Info:

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There is no required textbook.
I will scan all course notes and post them on my webpage:

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Your grade is based on

$\frac{1}{3}$ Weekly Homework (Fridays)
$\frac{1}{3}$ Weekly Quiz (Mondays)
$\frac{1}{3}$ Final Exam

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Course Topic: Discrete Mathematics

"Discrete" is the opposite of "Continuous" (Calculus)

This is particularly relevant for computer applications: any continuous problem must first be "discretized" before it can be implemented on a computer.

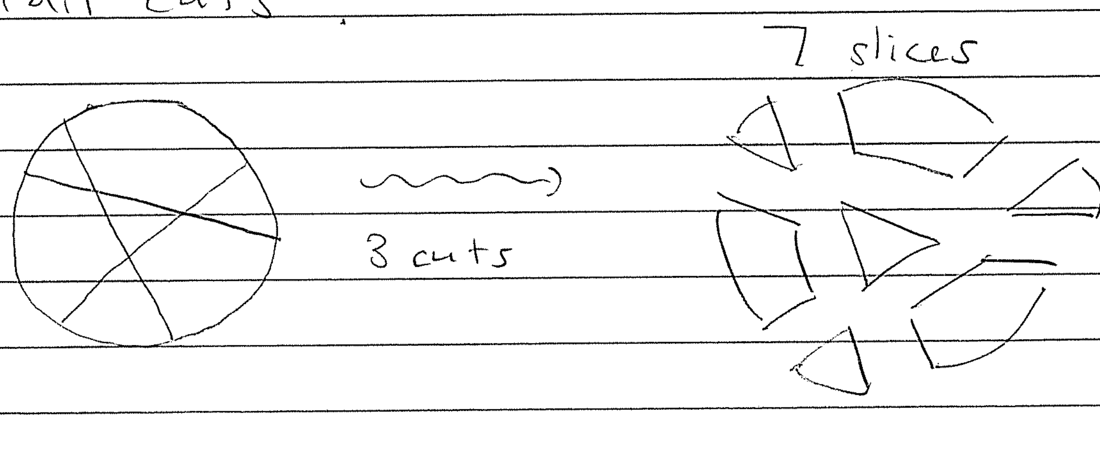
Some major themes:

- Recursion / Induction
- Counting and Probability
- Number Theory and Cryptography
- What Do You Want to See?

Before we begin, let's look at a sample problem.

Problem (Jacob Steiner, 1826):

What is the maximum number of slices of pizza you can get by making n strait cuts?



More formally, let

L_n := The maximum number of regions cut out by n (infinite) lines in the (infinite) plane.

[Remark: The symbol ":=" means "is <u>defined</u> to be".]

Our Goal: <u>Solve</u> for L_n.

What does that mean?

- Say how to compute L_n.
- Say how to compute L_n efficiently.
- Give a "formula" for L_n.
- Give a "nice formula" for L_n.

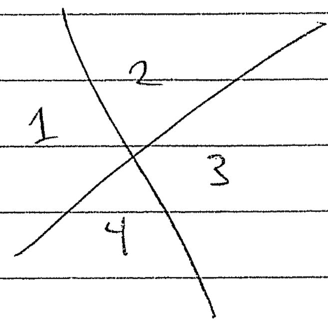
⋮

We'll see

To analyze a discrete problem we always start with <u>experiments</u>.

Obviously $L_0 = 1$

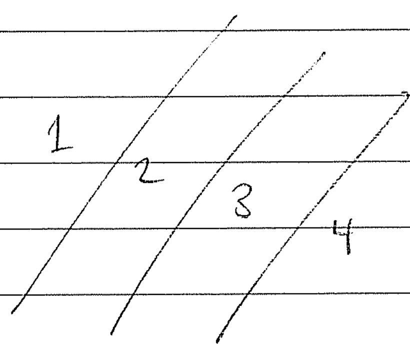
$L_1 = 2$.

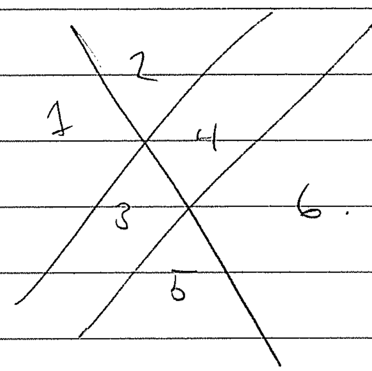


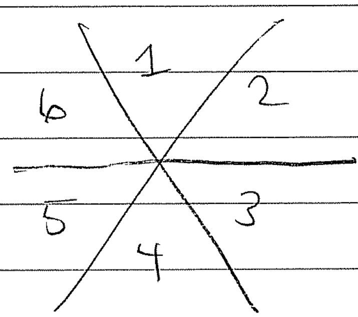
$L_2 = 4$.

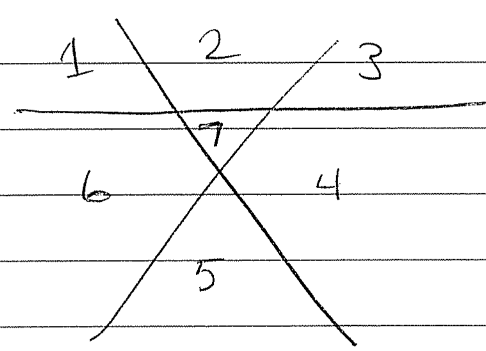
[Guess ("Conjecture"): $L_n = 2^n$?]

$L_3 = ?$









It looks like $L_3 = 7$

So our guess was <u>wrong</u>. :(

$$L_3 = 7 \neq 2^3 = 8$$

Now <u>you</u> do some experiments

(Wait)

Table of observations

n	0	1	2	3	4	5	6	...
L_n	1	2	4	7	11	16	22	...

Can we come up with a theory to explain these observations?

(Wait)

Here's the key observation:

$$\boxed{L_n = L_{n-1} + n}$$

for $n > 0$

There are two issues:

(1) Why is this true?

(2) Can we use it to solve the problem?

Let's do (2) first because it's more interesting.

We have a recurrence relation with an initital condition

- $L_0 = 1$
- $L_n = L_{n-1} + n$ for $n > 0$.

We can expand this out!

$$L_1 = L_0 + 1 = 1 + 1$$
$$L_2 = L_1 + 2 = 1 + 1 + 2$$
$$L_3 = L_2 + 3 = 1 + 1 + 2 + 3$$
$$L_4 = L_3 + 4 = 1 + 1 + 2 + 3 + 4$$

$$L_n = 1 + (1 + 2 + \cdots + n)$$

$$L_n = 1 + \sum_{k=1}^{n} k$$ ← the sum of k as k goes from 1 to n

Yay. We have a "formula" for L_n:

$$L_n = 1 + \sum_{k=1}^{n} k.$$

But is it a "good formula"?

(Maybe we can't do any better?)

This problem is nice because there's a lucky trick.

Let $S_n := \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n.$

Theorem (Carl Friedrich Gauss, when he was 6 years old):

$$S_n = \frac{n(n+1)}{2}$$

Proof: Consider the quantity $2 \cdot S_n$.

$$\begin{aligned} 2S_n &= S_n + S_n \\ &= (1 + 2 + 3 + \cdots + n) \\ &\quad + (n + n-1 + n-2 + \cdots + 1). \\ &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}}. \end{aligned}$$

So $2S_n = n(n+1)$

$$\Longrightarrow S_n = \frac{n(n+1)}{2}. \quad ///$$

This gives us a "closed formula" for L_n.

$$\begin{aligned} L_n &= 1 + S_n \\ &= 1 + \frac{n(n+1)}{2} \\ &= \frac{2+n(n+1)}{2} \\ &= \frac{n^2+n+2}{2}. \end{aligned}$$

Conclusion:

$$L_n = \frac{n^2+n+2}{2}$$

Is that a "nice" formula? Yes!

[Thinking Homework: Why is n^2+n+2 always an even number?]

Bonus: We get an asymptotic estimate. For large n we have

$$L_n \sim \frac{1}{2} n^2$$

[Meaning: $\lim_{n \to \infty} \frac{L_n}{\frac{1}{2} n^2} = 1$.]

That might be useful.

WE RAN OUT OF TIME HERE.

Now we return to 1.

Why is $L_n = L_{n-1} + n$? We want a "proof".

Imagine we have successfully divided the plane into L_n regions using n lines

$$\ell_1, \ell_2, \ell_3, \dots, \ell_n.$$

Now remove line ℓ_n. How many regions do we have just using the lines

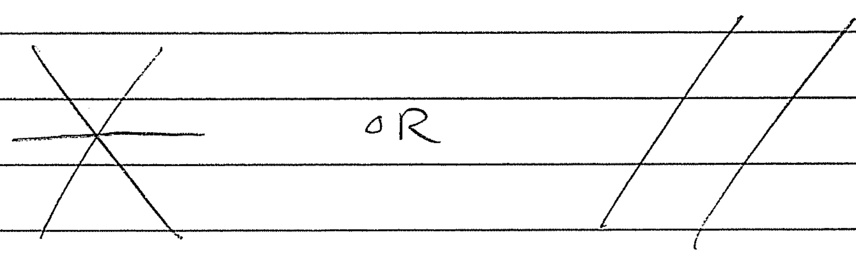
$$\ell_1, \ell_2, \dots, \ell_{n-1} \;?$$

Answer: L_{n-1} regions.

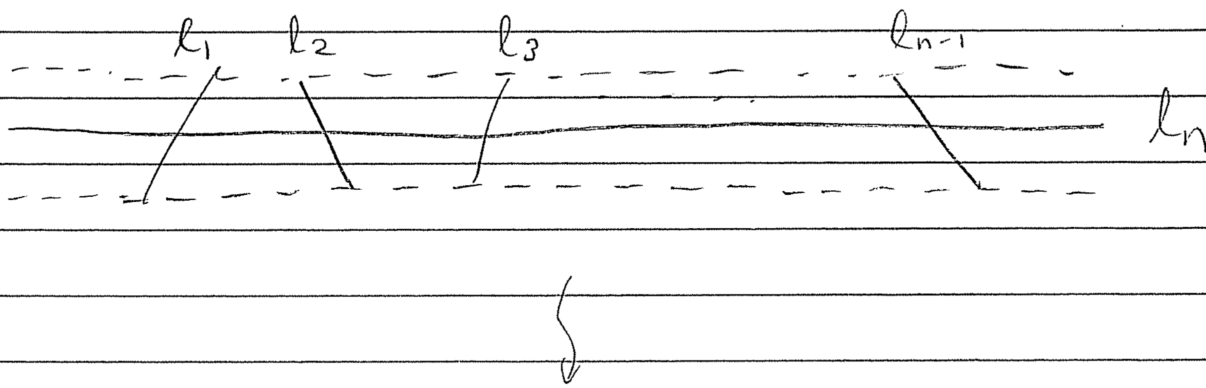
[Reason: We must get the maximum possible number of regions because L_n is by definition maximal. Don't worry too much about this.]

Okay, now what happens to these L_{n-1} regions when we add line ℓ_n back in?

Observe that we never have



(otherwise we don't get the maximum number of regions). So a small neighborhood of the line ℓ_n looks like

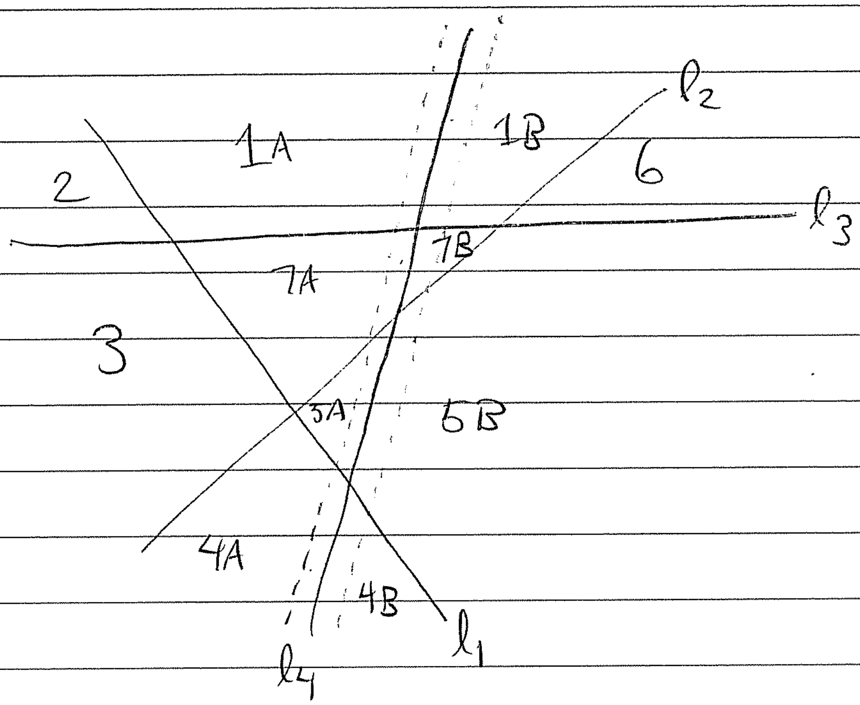


That is, line ℓ_n passes through exactly n of the original L_{n-1} regions and divides each of these n regions into 2. Thus we get n <u>new</u> regions:

$$\boxed{L_n = L_{n-1} + n}.$$

///

If you don't believe it, here's a picture:



Line ℓ_4 creates 4 new regions.

Problems for Thinking:

1. Explain why

$$\lim_{n \to \infty} \frac{L_n}{\frac{1}{2} n^2} = 1.$$

2. Is there a "closed formula" for the sum of the first n squares?

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = ?$$

3. Jacob Steiner (1826) also considered the problem of dividing 3-dim space by n 2-dim planes

Let P_n be the maximum number of 3-dim regions you can get.

- Do you think P_n has a closed formula?
- How would you approach the problem?
- Roughly how big is P_n?

- HW 1 due at the beginning of Friday's class
- Quiz 1 at the beginning of Monday's class.

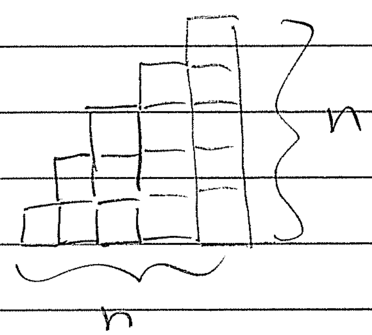
Last time we saw "Gauss' Trick".

Let $S_n := \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

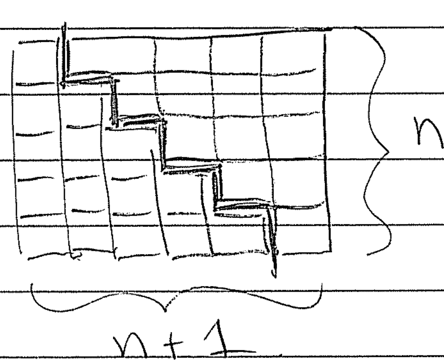
Then $S_n = \frac{n(n+1)}{2}$.

Proof: Observe that S_n is the area of a staircase made of 1×1 squares.

S_n = area of



Thus $2S_n$ is the area of two staircases put together.



This is a rectangle of area $n(n+1)$.

Hence $2S_n = n(n+1)$.

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2} \qquad //\!/$$

Now let's consider the sum of the first n squares.

Let $\square_n := \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$.

Q: Does this have a nice formula?

A: Yes, but it's harder to find.

Theorem: For all $n \geq 0$ we have

$$\square_n = \frac{n(n+1)(2n+1)}{6}$$

Proof (It's a trick; don't worry if you wouldn't think of this).

We will temporarily consider the sum of the first n cubes.

$$\boxplus_n := \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

Now we will look at the sum $\boxplus_{n+1}$ in two different ways.

On one hand we have

$$\boxplus_{n+1} = \sum_{k=1}^{n+1} k^3 = \underbrace{1^3 + 2^3 + \cdots + n^3} + (n+1)^3$$

$$= \sum_{k=1}^{n} k + (n+1)^3$$

$$= \boxplus_n + (n+1)^3 .$$

On the other hand we have

$$\boxplus_{n+1} = \sum_{k=1}^{n+1} k^3 = \sum_{k=0}^{n} (k+1)^3$$

$$= \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1).$$

$$= \sum_{k=0}^{n} k^3 + 3 \cdot \sum_{k=0}^{n} k^2 + 3 \sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$$= \sum_{k=1}^{n} k^3 + 3 \cdot \sum_{k=1}^{n} k^2 + 3 \cdot \sum_{k=1}^{n} k + \sum_{k=0}^{n} 1 .$$

$$= \boxplus_n + 3 \cdot \square_n + 3 \cdot S_n + (n+1)$$

Equating both expressions for $\boxplus_{n+1}$ gives

$$\cancel{\boxplus_n} + (n+1)^3 = \cancel{\boxplus_n} + 3 \cdot \square_n + 3 \cdot S_n + (n+1).$$

Note that $\boxplus_n$ cancels. (That's the trick!)

Now we can <u>solve</u> for $\square_n$:

$$(n+1)^3 = 3\square_n + 3S_n + (n+1)$$

$$3\square_n = (n+1)^3 - 3S_n - (n+1)$$

$$= (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)$$

$$= (n+1)\left((n+1)^2 - \frac{3n}{2} - 1\right)$$

$$= (n+1)\left(\frac{2(n+1)^2 - 3n - 2}{2}\right)$$

$$= \frac{1}{2}(n+1)\left(2n^2 + 4n + 2 - 3n - 2\right)$$

$$= \frac{1}{2}(n+1)\left(2n^2 + n\right)$$

$$= \frac{1}{2} n(n+1)(2n+1).$$

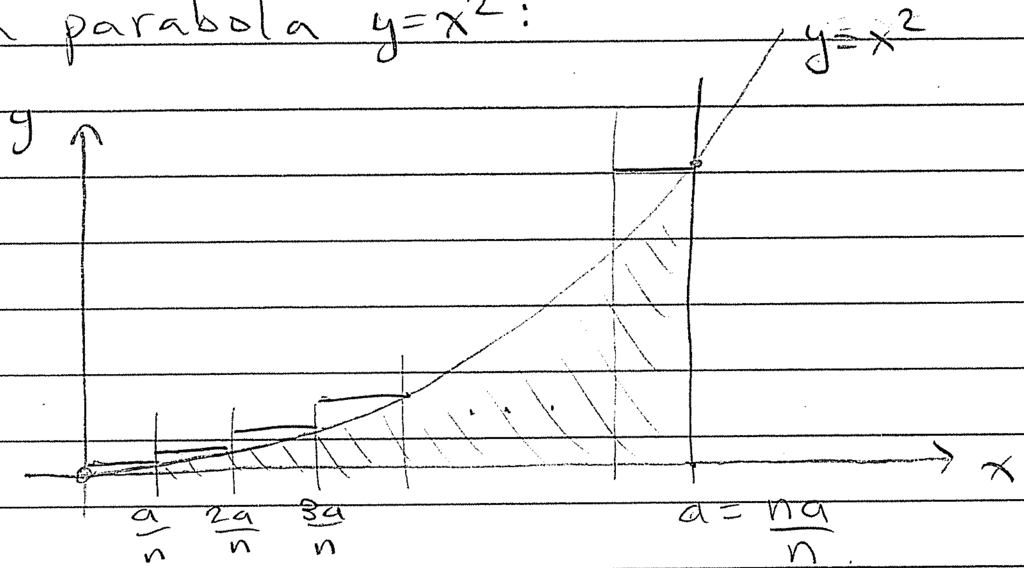
Hence

$$\square_n = \frac{1}{6} n(n+1)(2n+1)$$ ///

Q: Why would anyone care?

A: Here's why Pierre de Fermat cared (1637).

He wanted to compute the area under a parabola $y=x^2$:



$$\text{Exact area} = \int_0^a x^2\,dx$$

$$\approx \text{sum of areas of rectangles}$$

$$= \sum_{k=1}^{n} \underbrace{\frac{a}{n}}_{\text{base}} \underbrace{\left(\frac{ka}{n}\right)^2}_{\text{height}}$$

$$= \sum_{k=1}^{n} \frac{a}{n} \frac{k^2 a^2}{n^2}$$

$$= \frac{a^3}{n^3} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{a^3}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{a^3}{n^3} \left(\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \right)$$

$$= a^3 \left(\frac{1}{3} + \boxed{\frac{1}{2n} + \frac{1}{6n^2}} \right)$$

this goes to 0 as $n \to \infty$.

Taking the limit as $n \to \infty$ gives

$$\int_0^a x^2 \, dx = \lim_{n \to \infty} a^3 \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)$$

$$= a^3 \left(\frac{1}{3} + 0 + 0 \right)$$

$$= \frac{a^3}{3}.$$

Of course the Fundamental Theorem of Calculus would be faster, but it was not known in 1637.

Remarks:

1. Actually, Fermat only needed the asymptotic estimate

$$\square_n \sim \frac{1}{3} n^3$$

2. Consider the sum of pth powers:

$$f_p(n) := \sum_{k=1}^{n} k^p$$

Fermat proved that $f_p(n) \sim \frac{1}{p+1} n^{p+1}$

and hence $\int_0^a x^p \, dx = \frac{1}{p+1} a^{p+1}$.

3. We know $\square_n \sim \frac{1}{4} n^4$.

Can you give an <u>exact</u> formula?