Math 309 **Summer 2014**
Boolean Algebras **Drew Armstrong**

A **Boolean algebra** is a set B together with three functions, called

- **join** $\vee : B \times B \to B$
- **meet** $\wedge : B \times B \to B$
- **complement** $\neg : B \to B$,

and two special elements called $0, 1 \in B$, satisfying the following five rules (called "axioms"):

(1) **Associaive Properties.** For all $, a, b, c \in B$ we have

- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- $a \vee (b \vee c) = (a \vee b) \vee c$

(2) **Commutative Properties.** For all $a, b \in B$ we have

- $a \vee b = b \vee a$
- $a \wedge b = b \wedge a$

(3) **Properties of** 0 **and** 1**.** For all $a \in B$ we have

- $a \vee 0 = a$
- $a \wedge 1 = a$

(4) **Properties of Complement.** For all $a \in B$ we have

- $a \vee \neg a = 1$
- $a \wedge \neg a = 0$

(5) **Distributive Properties.** For all $a, b, c \in B$ we have

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

That's it. What are the advantages of this very abstract definition? There are at least two. The first advantage is that computers don't understand human things like Venn diagrams and logical arguments. The language of Boolean algebra is purely formal and easy to teach to a computer. The second advantage is that is allows us humans to make fewer mistakes by converting the analysis of arguments into the mechanical manipulation of symbols. (I've learned that students don't like to analyze arguments but they do like to manipulate symbols.)

Here's how it works. Beginning with the five axioms, we can begin to prove other properties (called "theorems"). We say that a **theorem** is any true equation that can be obtained by successively applying the axioms. Before proving a few theorems, I will state a useful general principle.

The Duality Principle. Note that the axioms of Boolean algebra remain the same if we simultaneously switch the symbols $\vee \leftrightarrow \wedge$ and the symbols $0 \leftrightarrow 1$. Therefore, any theorem obtained from the axioms will remain true if we make these switches. This will save time.

(6) **Theorem.** For all $a \in B$ we have

- $a \vee a = a$
- $a \wedge a = a$

Proof. In view of the Duality Principle, we only need to prove the first statement. Here is the proof. At each step I will quote the axiom used in the margin.

$$
\begin{aligned}
a &= a \vee 0 && (3)\\
&= a \vee (a \wedge \neg a) && (4)\\
&= (a \vee a) \wedge (a \vee \neg a) && (5)\\
&= (a \vee a) \wedge 1 && (4)\\
&= a \vee a && (3)
\end{aligned}
$$

□

(7) **Theorem.** We have

- $\neg 0 = 1$
- $\neg 1 = 0$

Proof. Again, we will just prove the first statement.

$$
\begin{aligned}
\neg 0 &= \neg 0 \vee 0 && (3)\\
&= 0 \vee \neg 0 && (2)\\
&= 1 && (4)
\end{aligned}
$$

□

(8) **Theorem.** For all $a \in B$ we have

- $a \vee 1 = 1$
- $a \wedge 0 = 0$

Proof.

$$
\begin{aligned}
a \vee 1 &= a \vee (a \vee \neg a) && (4)\\
&= (a \vee a) \vee \neg a && (1)\\
&= a \vee \neg a && (6)\\
&= 1 && (4)
\end{aligned}
$$

□

Notice that we used Theorem (6) in the proof of Theorem (8). That's okay. We're allowed to use anything that came before, as long as we don't create any circles.

(9) **Theorem (Absorption Properties).** For all $a, b \in B$ we have

- $a \vee (a \wedge b) = a$
- $a \wedge (a \vee b) = a$

Proof.

$$
\begin{aligned}
a \vee (a \wedge b) &= (a \wedge 1) \vee (a \wedge b) && (3)\\
&= a \wedge (1 \vee b) && (5)\\
&= a \wedge 1 && (2), (8)\\
&= a && (3)
\end{aligned}
$$

□

(10) **Theorem (Cancellation).** If for some $a, b, c \in B$ we have $a \wedge c = b \wedge c$ and $a \vee c = b \vee c$, it follows that $a = b$.

Proof. To begin, we assume that we have $a, b, c \in B$ such that $a \wedge c = b \wedge c$ and $a \vee c = b \vee c$. It follows that

$$\begin{aligned} a &= a \vee (a \wedge c) && (9) \\ &= a \vee (b \wedge c) && \text{by hypothesis} \\ &= (a \vee b) \wedge (a \vee c) && (5) \\ &= (a \vee b) \wedge (b \vee c) && \text{by hypothesis} \\ &= b \vee (a \wedge c) && (2), (5) \\ &= b \vee (b \wedge c) && \text{by hypothesis} \\ &= b && (9) \end{aligned}$$

□

(11) **Theorem (Uniqueness of Complements).** If for some $a, b \in B$ we have $a \wedge b = 0$ and $a \vee b = 1$, it follows that $b = \neg a$.

Proof. To begin, suppose that we have $a, b \in B$ such that $a \wedge b = 0$ and $a \vee b = 1$. From this we get

$$\begin{aligned} a \wedge b &= 0 && \text{by hypothesis} \\ &= a \wedge \neg a && (4) \end{aligned}$$

and

$$\begin{aligned} a \vee b &= 1 && \text{by hypothesis} \\ &= a \vee \neg a && (4) \end{aligned}$$

Finally, Theorem (10) implies that $b = \neg a$. (Why?) □

All of that was essentially preamble. Now comes the really important theorem.

(12) **Theorem (De Morgan's Identities).** For all $a, b \in B$ we have

- $\neg(a \vee b) = \neg a \wedge \neg b$
- $\neg(a \wedge b) = \neg a \vee \neg b$

Proof. By the Duality Principle we only need to prove the first statement $\neg(a \vee b) = \neg a \wedge \neg b$. By Theorem (11) it is enough to show that $(a \vee b) \wedge (\neg a \wedge \neg b) = 0$ and $(a \vee b) \vee (\neg a \wedge \neg b) = 1$. To establish these two equations, note that

$$\begin{aligned} (a \vee b) \wedge (\neg a \wedge \neg b) &= [(\neg a \wedge \neg b) \wedge a] \vee [(\neg a \wedge \neg b) \wedge b] && (2), (5) \\ &= [\neg b \wedge (a \wedge \neg a)] \vee [\neg a \wedge (b \wedge \neg b)] && (1), (2) \\ &= (\neg b \wedge 0) \vee (\neg a \wedge 0) && (4) \\ &= 0 \vee 0 && (8) \\ &= 0 && (3) \text{ or } (6) \end{aligned}$$

and

$$\begin{aligned}
(a \vee b) \vee (\neg a \wedge \neg b) &= [(a \vee b) \vee \neg a] \wedge [(a \vee b) \vee \neg b] && (5)\\
&= [(a \vee \neg a) \vee b] \wedge [(b \vee \neg b) \vee a] && (1), (2)\\
&= (1 \vee b) \wedge (1 \vee a) && (4)\\
&= 1 \wedge 1 && (2), (8)\\
&= 1 && (3) \text{ or } (6)
\end{aligned}$$

□

I went through all this trouble because De Morgan's Identities are extremely useful and I want to be able to use them with impunity. We could have convinced ourselves of these much more quickly using a Venn diagram or a truth table, so why did we bother will all this abstract algebra? Again, two reasons. First: this language allows us to tell De Morgan's Identities to a computer; the proof from the axioms guarantees that the computer's idea and our human idea will be the same, as long as we and the computer agree on the axioms. Second: we could use a Venn diagram or a truth table to **prove** the identities, but how would we **guess** them in the first place? Algebraic manipulation is a good way to generate new theorems without having to think too much (or at all, in the case of a computer).

Remark: I will never ask you to prove anything this involved. In practice, the formulas on these four pages are all we will ever need. So our tour of Boolean algebra is done.