7/25/14

HW 4 due now.
Quiz 4 on Monday

Let's discuss HW4.

Problem 3.

Here we show that

$$\text{"}(P \vee Q) \Rightarrow R\text{"} = \text{"}(P \Rightarrow R) \wedge (Q \Rightarrow R)\text{"}$$

In other words: The statement "if P or Q is true then R is true" is logically equivalent to the statement "if P is true then R is true, <u>and</u> if Q is true then R is true".

It sounds more awkward this way but it's easier to work with.

To prove $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ we just prove $P \Rightarrow R$ and $Q \Rightarrow R$ separately.

Problem 4: Prove that

"if m or n is even then mn is even".

<u>Proof</u>: Let $P =$ "m is even"
$Q =$ "n is even"
$R =$ "mn is even"

Then we want to prove

$$(P \vee Q) \Rightarrow R.$$

By Problem 3 this is the same as

$$(P \Rightarrow R) \wedge (Q \Rightarrow R).$$

In other words, "if m is even then mn is even, <u>and</u> if n is even then m is even".

We will prove them separately.

First, suppose that m is even so that $m = 2k$ for some $k \in \mathbb{Z}$.
In this case we have

$$mn = 2kn = 2(kn) = 2(\text{something}),$$

so that mn is even as desired.

Second, suppose that n is even so that $n = 2l$ for some $l \in \mathbb{Z}$. In this case we have

$$mn = m2l = 2(ml) = 2(\text{something})$$

so that mn is even as desired. We conclude that

$$(P \Rightarrow R) \wedge (Q \Rightarrow R)$$

is true and hence

$$(P \vee Q) \Rightarrow R$$

is true. //

Problem 5:

Let E = "is even"
O = "is odd"

Then we can make a table

m	n	mn	m+n
E	E	E	E
E	O	E	O
O	E	E	O
O	O	O	E

This table reminds me of

P	Q	$P \vee Q$	$P \oplus Q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	T

If we interpret P = "m is even" and Q = "n is even" then the 3rd columns say that

"mn is even" = "m is even OR n is even"

This is just what we proved in Problems 1–4.

The 4th columns say

"$m+n$ is even" = "m is even" $\oplus$ "n is even"

Can we say this better?

Use the disjunctive normal form

$$P \oplus Q = (P \wedge Q) \vee (\neg P \wedge \neg Q).$$

Then

"$m+n$ is even" = "m and n are both even, OR m and n are both odd"

That makes sense right?

Let's look at it another way:

+	E	O
E	E	O
O	O	E

×	E	O
E	E	E
O	E	O

We seem to have a "number system" with two "numbers" called E and O.

E = the property of being even
O = the property of being odd.

Did you know you could "add" and "multiply" such things? Well, you can.

Thinking Problem: Let

A = the property of being a multiple of 3
B = ... 1 more than a multiple of 3
C = ... 2 more than a multiple of 3.

These also form a "number system":

+	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

×	A	B	C
A	A	A	A
B	A	B	C
C	A	C	B

How would you make sense of this?

Recall the

☆ Principle of Induction:

Consider a function $P : \mathbb{N} \to \{T, F\}$.
If

(1) $P(b) = T$ for some $b \in \mathbb{N}$

and

(2) $P(k) \Rightarrow P(k+1)$ for all $k \geq b$

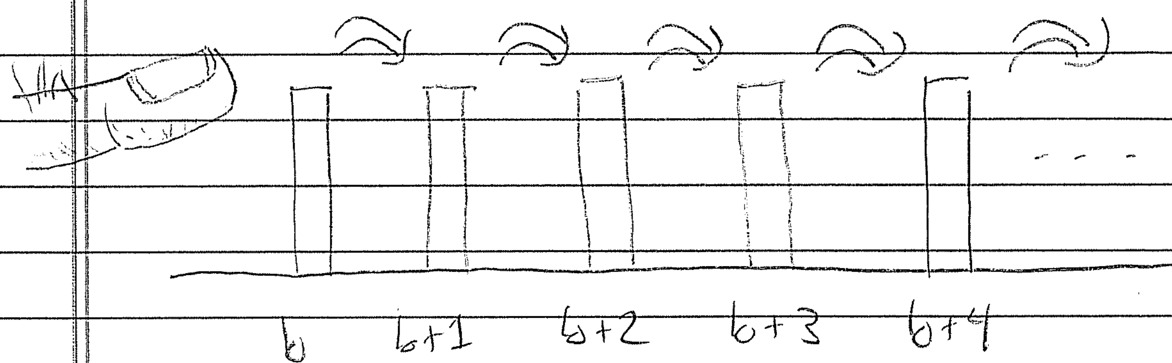
then we agree to say that

$$P(n) = T \text{ for } \underline{\text{all}}\ n \geq b.$$

Last time I gave an analogy of a computer trying to verify that $P(n) = T$ for all $n \geq b$ and breaking down because of the 2nd law of thermodynamics or some such thing.

Here's a different analogy:

induction $\equiv$ dominoes



(1) Your finger.

(2) The force of gravity

Both are necessary to this process.

Here's a cautionary example:

We say that a set of horses is monochromatic if all the horses in the set have the same color.

Theorem:

Every (finite) set of horses is monochromatic. (In other words, all horses have the same color.)

Proof by Induction:

Given $n \in \mathbb{N}$ let

$P(n) =$ "Every set of n horses is monochromatic"

First we verify the base case. Clearly every set of 1 horse is monochromatic, so

$$P(1) = T.$$

Next we verify the induction step. Assume (hypothetically) that $P(k) = T$, i.e., every group of k horses is monochromatic. In this case we want to show that $P(k+1) = T$.

So consider any set S of $k+1$ horses and consider any two horses $x, y \in S$. We will show that x and y have the same color.

To do this, let $z \in S$ be any third horse.

Since the set $S - \{y\}$ has size k we know by assumption that $S - \{y\}$ is monochrom. Then since $x, z \in S - \{y\}$ we know that x & z have the same color.

Similary, we know that $S - \{x\}$ is monochrom. Then since $y, z \in S - \{x\}$ we know that y & z have the same color.

By transitivity we conclude that x & y have the same color. Since this is true for any $x, y \in S$ we conclude that S is monochromatic. Since this is true for any set S of $k+1$ horses we conclude that

$P(k+1) = T$ as desired.

We have thus proved that

$$P(k) \Rightarrow P(k+1).$$

By the principle of induction we conclude that $P(n) = T$ for all $n \geq 1$.

In other words, all horses have the same color.

OK, so clearly we made a mistake, but what EXACTLY was the mistake ??

We successfully showed that

(1) $P(1) = T$

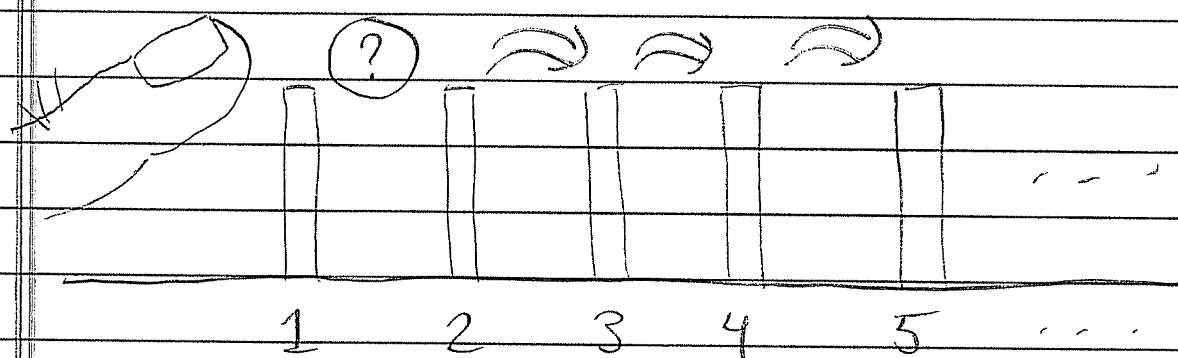
and

(2) For all $k \geq 2$ we have

$$P(k) \Rightarrow P(k+1).$$

(Yes, our argument that $P(k) \Rightarrow P(k+1)$ implicitly used the assumption that $k \geq 2$ when we said "let $z \in S$ be any third horse". What if there is no third horse? Then our argument falls apart.)

Here's the situation:



The finger is OK but there is a small problem with gravity; namely, there is NO GRAVITY between dominoes 1 and 2. So only domino 1 will fall down. The rest remain standing.

If we could somehow prove $P(2) = T$ then the rest would fall down.

So close, and yet so far...

7/28/14

Quiz 4 now (20 minutes)

This week: Induction.

Recall the

☆ Principle of Induction:

Consider a function $P : \mathbb{N} \to \{T, F\}$.
If

① $P(b) = T$ for some $b \in \mathbb{N}$

and

② for all $k \geq b$ we have

$$P(k) \Longrightarrow P(k+1)$$

then we agree to say that

$$P(n) = T \text{ for \underline{all} } n \geq b.$$

///

Using induction takes practice, so this week we will practice.

<u>Definition</u>: Given integers $n, d \in \mathbb{Z}$ we define the statement

$$\text{"}d \mid n\text{"} := \text{"}\exists k \in \mathbb{Z};\ n = dk\text{"}$$

We read "$d \mid n$" as "d divides n" or "n is divisible by d".

<u>Problem</u>: Prove that for all $n \in \mathbb{N}$ we have

$$6 \mid (2n^3 + 3n^2 + n).$$

<u>Proof</u>: For all $n \in \mathbb{N}$ define the statement

$$P(n) := \text{"}6 \mid (2n^3 + 3n^2 + n)\text{"}.$$

Base Case:

$$P(0) = \text{"}6 \mid (2 \cdot 0^3 + 3 \cdot 0^2 + 0)\text{"} = \text{"}6 \mid 0\text{"}$$

Is this true?

Yes. Recall that

$$\text{"}6 \mid 0\text{"} = \text{"}\exists k \in \mathbb{Z},\ 0 = 6k\text{"}.$$

This is true because we can take $k = 0$.

[You should check a few more cases, $P(1)$, $P(2)$, $P(3)$ just to make sure you believe the result, but it's not strictly necessary for the proof.]

Induction Step: Consider any $k \geq 0$ and "assume for induction" that $P(k) = T$, i.e., assume that there exists $d \in \mathbb{Z}$ such that

$$2k^3 + 3k^2 + k = 6d.$$

In this case we want to show that $P(k+1) = T$, i.e., that

$$6 \mid \left[2(k+1)^3 + 3(k+1)^2 + (k+1)\right]$$

OK, now what?

Probably we should expand.

$$2(k+1)^3 + 3(k+1)^2 + (k+1)$$

$$= 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + (k+1)$$

$$= \cancel{2k^3} + \cancel{6}k^2 + \cancel{6}k + 2 + \cancel{3k^2} + \cancel{6}k + 3 + \cancel{k} + 1$$

$$= 2k^3 + 9k^2 + 13k + 6$$

OK, now what?

Somehow we must use the fact that

$$2k^3 + 3k^2 + k = 6d \ldots\ldots$$

I guess we went too far. Back up.

$$2(k+1)^3 + 3(k+1)^2 + (k+1)$$

$$= \underbrace{\left(2k^3 + 3k^2 + k\right)}_{} + 6k^2 + 12k + 6$$

$$= 6d + 6k^2 + 12k + 6$$

$$= 6(d + k^2 + 2k + 1) = 6\,(\text{something}).$$

We conclude that

$$P(k+1) = \text{"}6 \,\Big|\, \left[2(k+1)^3 + 3(k+1)^2 + (k+1)\right]\text{"} = T$$

as desired. In summary we have shown that for all $k \geq 0$ we have

$$P(k) \Rightarrow P(k+1).$$

End of induction step.

By the principle of Induction we conclude that

$$P(n) = T \text{ for all } n \geq 0.$$

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[Thinking Problem: In fact, it is true that

$$6 \mid (2n^3 + 3n^2 + n)$$

for all $n \in \mathbb{Z}$ (including negative n). How would you prove this?]

Another Problem: Let

$F_n :=$ The set of binary strings of length n in which no two 1's are consecutive.

Find a formula for $\#F_n$.

Experiment:

$F_0 = \{\emptyset\}$

$F_1 = \{0, 1\}$

$F_2 = \{00, 01, 10\}$

$F_3 = \{000, 100, 010, 001, 101\}$

$F_4 = ?$

0000	1010
1000	1001
0100	0101
0010	That's All.
0001	

Define $f_n := \# F_n$. We have

n	0	1	2	3	4	5
f_n	1	2	3	5	8	...

Can you guess a formula yet?
If not, we need more data.

n	4	5	6	7	8	9	10	...
f_n	8	13	21	34	55	89	144	...

Can you guess a formula yet? No.

OK, but maybe we can see a pattern or some structure?

Eventually we will observe the following fact: For all $n \geq 2$ we have

$$\boxed{f_n = f_{n-1} + f_{n-2}}$$

<u>Why</u> is this true?

Proof: We can write F_n as a disjoint union of two sets

$$F_n = A \sqcup B, \quad \text{where}$$

$$A = \{ x \in F_n : x \text{ begins with } 0 \}$$

$$B = \{ x \in F_n : x \text{ begins with } 1 \}.$$

I claim that

$$\# A = f_{n-1}.$$

Indeed, if the first symbol is 0, then the rest of the word is an element of F_{n-1}

$$0 \; \underbrace{\boxed{\text{an element of } F_{n-1}}}_{\text{length } n-1} \in A.$$

We get a 1-1 correspondence $A \leftrightarrow F_{n-1}$ and hence $\# A = \# F_{n-1} = f_{n-1}$.

I also claim that

$$\# B = f_{n-2}.$$

Indeed, if the first symbol is 1 then the second symbol *must* be 0 (since there are no consecutive 1's. Then the rest of the word is an element of F_{n-2}

$$\underline{1}\ \underline{0}\ \underbrace{\boxed{\text{an element of } F_{n-2}}}_{\text{length } n-2} \in B$$

We get a 1-1 correspondence $B \leftrightarrow F_{n-2}$ and hence $\# B = \# F_{n-2} = f_{n-2}$.

We conclude that

$$\begin{aligned} f_n = \# F_n &= \# A + \# B \\ &= f_{n-1} + f_{n-2} \end{aligned}$$

//

Example:

$$F_4 = \left\{ \begin{array}{cc} 0000 & 1000 \\ 0100 & 1010 \\ 0010 & 1001 \\ 0001 & \\ 0101 & \end{array} \right\}$$

$$f_4 = f_3 + f_2$$

$$8 = 5 + 3$$

OK great. Does that help us find a formula?

Maybe not, but if we can <u>guess</u> a formula, this will help us <u>prove</u> it.

I will give you the guess for free!

Let $\alpha := \dfrac{1+\sqrt{5}}{2}$ and $\beta := \dfrac{1-\sqrt{5}}{2}$

Then (Guess): For all $n \geq 0$ we have

$$f_n = \frac{1}{\sqrt{5}}\left(\alpha^{n+2} - \beta^{n+2}\right)$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$$

Can you prove this using induction?

7/29/14

HW 5 due Friday
Quiz 5 on Monday
Wed 8/6 last day of class
Thurs 8/7 reading day
Fri 8/8 final exam

Today: More induction.

Last time we considered a problem:

Let $F_n :=$ The set of binary strings of length n in which no two 1's are consecutive.

Let $f_n := \# F_n$.

Find a "formula" for f_n.

Last time we found some data

n	0	1	2	3	4	5	6	...
f_n	1	2	3	5	8	13	21	...

We guessed the "recurrence formula"

$$f_n = f_{n-1} + f_{n-2},$$

then we proved it.

But we would still like a "closed formula"

$$f_n = ?$$

Two separate issues:

1. Can we guess a formula?

2. Given a proposed formula, can we prove it?

Let's skip issue 1 for now. I'll just tell you the formula and then we'll prove it.

☆ I claim that for all $n \in \mathbb{N}$ we have

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$$

That's pretty surprising, so we should check it before we try to prove it.

$n = 0$.

$$f_0 \stackrel{?}{=} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^2}{4} - \frac{(1-\sqrt{5})^2}{4}\right]$$

$$= \frac{1}{4\sqrt{5}}\left[(\cancel{1}+2\sqrt{5}+\cancel{5}) - (\cancel{1}-2\sqrt{5}+\cancel{5})\right]$$

$$= \frac{1}{\cancel{4\sqrt{5}}}(\cancel{4\sqrt{5}}) = 1. \quad \checkmark$$

$n = 1$.

$$f_1 \stackrel{?}{=} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^3}{8} - \frac{(1-\sqrt{5})^3}{8}\right]$$

$$\vdots$$

$$= \frac{1}{8\sqrt{5}} \left[\left(1+\sqrt{5}\right)^3 - \left(1-\sqrt{5}\right)^3 \right]$$

$$= \frac{1}{8\sqrt{5}} \Big[\left(\cancel{1} + 3\sqrt{5} + \cancel{3\cdot 5} + 1\cdot 5\sqrt{5}\right) - \left(\cancel{1} - 3\sqrt{5} + \cancel{3\cdot 5} - 1\cdot 5\sqrt{5}\right) \Big]$$

$$= \frac{1}{8\sqrt{5}} \left(3\sqrt{5} + 5\sqrt{5} + 3\sqrt{5} + 5\sqrt{5} \right)$$

$$= \frac{1}{8\sqrt{5}} \left(16\sqrt{5} \right) = 2 \quad \checkmark$$

Wow, I really don't want to check any more.

Can we just say we believe it now?

Good. Now let's try to prove it by induction. I recommend that we hide the details inside some convenient notation.

Consider the quadratic equation

$$x^2 - x - 1 = 0.$$

Its solutions are

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Let $\alpha := \frac{1+\sqrt{5}}{2}$ and $\beta := \frac{1-\sqrt{5}}{2}$.

[Remark : α is called the "golden ratio".]

By definition we have

$$\alpha^2 - \alpha - 1 = 0 \quad \text{and} \quad \beta^2 - \beta - 1 = 0$$
$$\alpha^2 = \alpha + 1 \qquad\qquad \beta^2 = \beta + 1.$$

This will be useful for hiding details.

Now we claim that

$$f_n = \frac{1}{\sqrt{5}} \left[\alpha^{n+2} - \beta^{n+2} \right].$$

What's the induction step?

Assume that $f_k = \frac{1}{\sqrt{5}}\left[\alpha^{k+2} - \beta^{k+2}\right]$.

We want to show $f_{k+1} = \frac{1}{\sqrt{5}}\left[\alpha^{k+3} - \beta^{k+3}\right]$.

OK, let's see. We have

$$\frac{1}{\sqrt{5}}\left[\alpha^{k+3} - \beta^{k+3}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{k+1}\alpha^{2} - \beta^{k+1}\beta^{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{k+1}(\alpha+1) - \beta^{k+1}(\beta+1)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{k+2} + \alpha^{k+1} - (\beta^{k+2} + \beta^{k+1})\right]$$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{k+2} - \beta^{k+2}\right] + \frac{1}{\sqrt{5}}\left[\alpha^{k+1} - \beta^{k+1}\right]$$

$$= f_k + f_{k-1} \quad ??$$

Do we know this?

Well, we assumed that

$$f_k = \frac{1}{\sqrt{5}}\left[\alpha^{k+2} - \beta^{k+2}\right].$$

Why don't we also assume that

$$f_{k-1} = \frac{1}{\sqrt{5}}\left[\alpha^{k+1} - \beta^{k+1}\right] ?$$

Then, assuming these two facts, we get

$$\frac{1}{\sqrt{5}}\left[\alpha^{k+3} - \beta^{k+3}\right]$$

$$= f_k + f_{k-1} \quad \text{by assumption}$$

$$= f_{k+1} \quad \text{by the recurrence we proved.}$$

I guess that does it,

but we're going to need a new legal contract for the extra assumption we made.

★ Principle of Strong Induction:

Consider a function $P : \mathbb{N} \to \{T, F\}$. If

1. $P(b) = P(b+1) = \cdots = P(b+d-1) = T$

and 2. For all $k \geq b$ we have

$(P(k) \wedge P(k+1) \wedge \cdots \wedge P(k+d-1)) \Longrightarrow P(k+d)$

then we agree to say that

$$P(n) = T \quad \forall\, n \geq b.$$

(Please sign here.)

In essence, we are allowed to assume the d previous cases, as long as we check d base cases.

The usual Principle of Induction corresponds to $d = 1$.

Why are we allowed to do this?

Let WO = Well-Ordering Axiom
PI = Principle of Induction
PSI = Principle of Strong Induction.

If you gave me enough time, I could prove to you that

$$\begin{tikzcd} & WO \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\ PI \arrow[rr, Leftrightarrow] & & PSI \end{tikzcd}$$

(but you probably wouldn't like it)

They are all logically equivalent. So you can just choose the one that's most convenient in any given situation.

Usually people aren't even explicit about this. They just say "by induction", and leave it to the reader to figure out the details.

Finally, let's write a nice proof.

Theorem: For all $n \geq 0$ we have

$$f_n = \frac{1}{\sqrt{5}}\left[\alpha^{n+2} - \beta^{n+2}\right].$$

Proof by induction: Let

$$P(n) = \text{“} f_n = \frac{1}{\sqrt{5}}\left[\alpha^{n+2} - \beta^{n+2}\right] \text{”}$$

We want to show that $P(n) = T \ \forall n \geq 0$.

Base Cases: We previously checked that $P(0) = P(1) = T$.

Induction Step: Assume for induction that $P(k) = P(k-1) = T$. [We are allowed to assume two cases because we checked two base cases.]
In this case we want to show that $P(k+1) = T$, in other words,

$$f_{k+1} = \frac{1}{\sqrt{5}}\left[\alpha^{k+3} - \beta^{k+3}\right].$$

Indeed, we saw previously that

$$\frac{1}{\sqrt{5}}\left[\alpha^{k+3}-\beta^{k+3}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\alpha^{k+2}-\beta^{k+2}\right]+\frac{1}{\sqrt{5}}\left[\alpha^{k+1}-\beta^{k+1}\right].$$

Since we assumed $P(k)=P(k-1)=T$ this means that

$$\frac{1}{\sqrt{5}}\left[\alpha^{k+3}-\beta^{k+3}\right]$$

$$= f_k + f_{k-1}$$ by assumption

$$= f_{k+1}$$ by the recurrence we proved.

We conclude that $P(k+1)=T$.

By induction we conclude that

$$P(n)=T \quad \forall n \geq 0.$$

//

So the formula is true.

But that still doesn't explain how anyone would guess the formula in the first place.

Q: So how could we <u>guess</u> the formula?

A: Well, this is harder.

The <u>best</u> way to do it is by linear algebra.

We write the recurrence as

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ f_{n-1} &= f_{n-1} \end{aligned}$$

and then express this via matrices

$$\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix}.$$

Now "diagonalize" this matrix.

If you don't know linear algebra, then I suppose we could use Calculus.

The trick is to define the "generating function" for the numbers f_n:

$$F(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + \cdots$$
$$= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

$$F(x) = \sum_{n \geq 0} f_n x^n$$

Turn the recurrence into information about $F(x)$:

$$f_n = f_{n-1} + f_{n-2}$$

$$\sum_{n \geq 2} f_n x^n = \sum_{n \geq 2} (f_{n-1} + f_{n-2}) x^n$$

$$F(x) - (1 + 2x) = \sum_{n \geq 2} f_{n-1} x^n + \sum_{n \geq 2} f_{n-2} x^n$$

$$= x \sum_{n \geq 2} f_{n-1} x^{n-1} + x^2 \sum_{n \geq 2} f_{n-2} x^{n-2}$$

$$= x \sum_{n\geq 1} f_n x^n + x^2 \sum_{n\geq 0} f_n x^n$$

$$= x\left(F(x) - 1 \right) + x^2 F(x).$$

Now solve for $F(x)$:

$$F(x) - (1+2x) = xF(x) - x + x^2 F(x).$$

$$F(x) - xF(x) - x^2 F(x) = 1 + 2x - x.$$

$$F(x)\left(1 - x - x^2\right) = 1 + x$$

$$F(x) = \frac{1+x}{1-x-x^2}.$$

We conclude that

$$\frac{1+x}{1-x-x^2} = f_0 + f_1 x + f_2 x^2 + \cdots$$

That is, f_n is the coefficient of x^n in the Taylor series for $(1+x)/(1-x-x^2)$ near $x = 0$.

OK, so compute the Taylor series.

How? Well, we'll try to be smart about it. First we observe that

$$1-x-x^2 = (1-\alpha x)(1-\beta x)$$

where α, β are as before. Then we use the method of partial fractions

$$\frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$$\frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A(1-\beta x)+B(1-\alpha x)}{(1-\alpha x)(1-\beta x)}$$

Equating numerators gives

$$\begin{aligned} 1+x &= A(1-\beta x) + B(1-\alpha x) \\ &= (A+B) + (-\beta A - \alpha B)x. \end{aligned}$$

Equating coefficients gives

$$\left.\begin{array}{l} A + B = 1 \\ -\beta A - \alpha B = 1 \end{array}\right\}$$

Solving this equation gives

$$A = \frac{-\alpha - 1}{\beta - \alpha} = -\frac{\alpha^2}{\sqrt{5}}$$

$$B = \frac{\beta + 1}{\beta - \alpha} = -\frac{\beta^2}{\sqrt{5}}$$

Finally we expand using geometric series.

$$F(x) = \frac{1+x}{1-x-x^2} = \frac{\alpha^2}{\sqrt{5}} \frac{1}{(1-\alpha x)} - \frac{\beta^2}{\sqrt{5}} \frac{1}{(1-\beta x)}$$

$$= \frac{\alpha^2}{\sqrt{5}} \left(1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots \right)$$

$$- \frac{\beta^2}{\sqrt{5}} \left(1 + \beta x + \beta^2 x^2 + \beta^2 x^3 + \cdots \right)$$

$$= \frac{1}{\sqrt{5}} \left((\alpha^2 - \beta^2) + (\alpha^3 - \beta^3) x + (\alpha^4 - \beta^4) x^2 + \cdots \right).$$

In other words, the coefficient of x^n in $F(x)$ is

$$\frac{1}{\sqrt{5}}\left[\alpha^{n+2} - \beta^{n+2}\right].$$

Aren't you glad you remembered how to compute Taylor series?

7/30/14

HW 5 due Friday
Quiz 5 Monday

Wed 8/6	Thurs 8/7	Fri 8/8
last day	reading day (lunch?)	exam

Today: More about Fibonacci.

- Leonardo of Pisa (a.k.a. Fibonacci) lived c.1170 – c.1250.

- Published "Liber Abaci" (1202) introducing Indian-Arabic numerals and decimal notation to Europe.

  0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

- Prior to that, Europeans used the much more primitive Roman numerals.

- But why is his name famous? Because of a silly problem about rabbits.

  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.

We defined the "Fibonacci sequence" as follows:

- $f_0 = 1$
- $f_1 = 2$
- $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

With some work, we used this recurrence and initial conditions to show that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$$

This is called "Binet's formula" (1843), even though it was known earlier to Euler and others.

What about the asymptotics?

Let $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$

$$\psi = \frac{1-\sqrt{5}}{2} = 1-\varphi$$

$$\approx -0.618\cdots$$

Then Binet's formula is

$$f_n = \frac{1}{\sqrt{5}}\left[\varphi^{n+2} - \psi^{n+2}\right]$$

But note that $|\psi| < 1$ and so

$$\lim_{n \to \infty} \psi^{n+2} = 0 .$$

This means that

$$f_n \sim \frac{1}{\sqrt{5}} \varphi^{n+2} \quad \text{as } n \to \infty$$

We could have guessed this without much work. Here's how:

Consider the ratios of consecutive Fibonacci numbers

n	0	1	2	3	4	5
$\frac{f_{n+1}}{f_n}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{13}$
	2	1.5	$1.6\overline{6}$..	1.6	1.625	$1.\overline{615384}$....

This seems to converge to something.

Let $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = l$.

We can compute l using the recurrence.

$$f_{n+1} = f_n + f_{n-1}$$

$$\frac{f_{n+1}}{f_n} = \frac{f_n}{f_n} + \frac{f_{n-1}}{f_n}$$

$$\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n}. \qquad (*)$$

As $n \to \infty$, $f_{n+1}/f_n \to l$.

What about $f_{n-1}/f_n \to$? Well,

$$\frac{f_{n-1}}{f_n} = \frac{1}{f_n/f_{n-1}} \longrightarrow \frac{1}{l}$$

because $\frac{f_n}{f_{n-1}} \to l$ as $n \to \infty$.

Taking limits on both sides of (*) gives

$$\ell = 1 + \frac{1}{\ell}$$

$$\ell^2 = \ell + 1$$

$$\ell^2 - \ell - 1 = 0.$$

$$\Longrightarrow \ell = \frac{1 \pm \sqrt{5}}{2}$$

Is it + or − ? Well, clearly $\ell > 0$ but $(1-\sqrt{5})/2 < 0$.

Hence $\ell = (1+\sqrt{5})/2 = \varphi$.

"The ratio of consecutive Fibonacci numbers tends to the golden ratio."

This leads us to guess that

$$f_n \sim C\varphi^{n+k}$$

for some constants C and k.

$\downarrow$

Indeed, then we have

$$\frac{f_{n+1}}{f_n} \sim \frac{\cancel{c}\,\varphi^{n+1+k}}{\cancel{c}\,\varphi^{n+k}} = \varphi^{n+1+k-(n+k)} = \varphi$$

as expected.

Note that this is independent of the intial conditions. The sequence

$$1, 3, 4, 7, 11, 18, 29, 47, \ldots$$

has the same property. To compute the constants C, k we must take the initial conditions into account, and this might be hard.

Here's another bit of fun. Consider the equation

$$\varphi = 1 + \frac{1}{\varphi}$$

Why don't we substitute this equation into itself?

$$\varphi = 1 + \frac{1}{\varphi}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{\varphi}}$$

and again,

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\varphi}}}$$

We can continue forever to get

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

This equation is true if we regard the RHS as a limit. This is called the "continued fraction expansion" of φ. It is an alternative to decimal notation.

$\int$

Another Example:

$$e = 2.718281828459 0 \ldots$$

has continued fraction expansion

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{\ddots}}}}}}}}}$$

We use the concise notation

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$$

This is nicer than the decimal expansion, because the decimal expansion has no pattern. In this language,

$$\varphi = [1; 1, 1, 1, 1, 1, 1, \ldots]$$

A bit more fun. Let's continue the Fibonacci sequence to the <u>left</u>.

n	-4	-3	-2	-1	0	1	2	3	4	5
f_n	1	-1	0	1	1	2	3	5	8	13

Continuing

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
f_n	-13	8	-5	3	-2	1	-1	0	1	1

Do you have a conjecture?

Well I have a suggestion. Can we please "re-index" the Fibonacci numbers by using different initial conditions

$$\hat{f}_0 = 0 \text{ and } \hat{f}_1 = 1 \quad ?$$

Thank you.

Now this looks better:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$\hat{f}_n$	5	-3	2	-1	1	0	1	1	2	3	5	8	13

Now we can make a conjecture:

$$\hat{f}_{-n} = (-1)^{n+1} \hat{f}_n .$$

Can we prove it? Yes.

<u>Proof</u>: Let $P(n) =$ " $\hat{f}_{-n} = (-1)^{n+1} \hat{f}_n$ ".

We want to show that $P(n) = T \; \forall n \geq 0$.

At least <u>two</u> base cases are true. ✓

Now assume for induction that $P(k)$ and $P(k-1)$ are true. In this case we want to show that $P(k+1)$ is also true.

First note that

$$\hat{f}_{-(k-1)} = \hat{f}_{-k+1}$$

$$= \hat{f}_{-k} + \hat{f}_{-k-1} \quad \text{by the defining recurrence,}$$

$$= \hat{f}_{-k} + \hat{f}_{-(k+1)}$$

Then solving for $\hat{f}_{-(k+1)}$ gives

$$\hat{f}_{-(k+1)} = -\hat{f}_{-k} + \hat{f}_{-(k-1)}$$

Now we use our assumptions

$$\hat{f}_{-k} = (-1)^{k+1}\hat{f}_{k}$$

$$\hat{f}_{-(k-1)} = (-1)^{k}\hat{f}_{k-1}$$

to obtain

$$\begin{aligned}
\hat{f}_{-(k+1)} &= -\hat{f}_{-k} + \hat{f}_{-(k-1)} \\
&= -(-1)^{k+1}\hat{f}_{k} + (-1)^{k}\hat{f}_{k-1} \\
&= (-1)^{k+2}\hat{f}_{k} + (-1)^{k+2}\hat{f}_{k-1} \\
&= (-1)^{k+2}\left[\hat{f}_{k} + \hat{f}_{k-1}\right] \\
&= (-1)^{k+1}\hat{f}_{k+1}.
\end{aligned}$$

The last step again uses the defining recurrence.

We have shown that if $P(k)$ and $P(k-1)$ are true then so is $P(k+1)$.

By induction we conclude that

$$P(n) = T \quad \forall n \geq 0.$$

as desired. //

7/31/14

HW 5 due tomorrow
Quiz 5 Monday

Next Week:
Wed last day of class
Thurs reading day
(lunch 2pm at Moon Thai & Japanese ?)
Friday final exam

Last time I used language like

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

This is called a continued fraction.
It is more convenient to write it as

$$\varphi = [\underline{1}; 1, 1, 1, 1, \ldots]$$

Note that this is an alternative to the decimal expansion

$$\varphi = 1.61803398875\ldots$$

Problem: Find the continued fraction expansion of $\sqrt{2}$.

$$\sqrt{2} = 1.41421\ldots$$

$$= 1 + 0.41421\ldots$$

$$= 1 + \frac{1}{2.41421\ldots}$$

$$= 1 + \frac{1}{2 + 0.41421\ldots}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2.41421\ldots}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

So I guess

$$\sqrt{2} = [1; 2, 2, 2, 2, \ldots].$$

Check: Assume the continued fraction converges to ℓ. Then we have

$$\ell = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

$$\ell = 1 + \frac{1}{1+\ell}.$$

$$(1+\ell)\ell = 1 + \ell + 1$$

$$\cancel{\ell} + \ell^2 = 2 + \cancel{\ell}$$

$$\ell^2 = 2$$

$$\ell = \pm\sqrt{2}$$

Obviously $\ell > 0$, so $\ell = +\sqrt{2}$

We can even turn this into an <u>algorithm</u> to compute $\sqrt{2}$.

Start with $x_0 = 1$ and then recursively define

$$x_{n+1} := 1 + \frac{1}{1+x_n}.$$

So $x_1 = 1 + \frac{1}{1+1} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$$x_2 = 1 + \frac{1}{1+\frac{3}{2}} = 1 + \frac{1}{5/2} = 1 + \frac{2}{5} = \frac{7}{5} = 1.4$$

$$x_3 = 1 + \frac{1}{1+\frac{7}{5}} = 1 + \frac{1}{12/5} = 1 + \frac{5}{12} = \frac{17}{12} = 1.41666\ldots$$

$$x_4 = 1 + \frac{1}{1+\frac{17}{12}} = 1 + \frac{1}{29/12} = 1 + \frac{12}{29} = \frac{41}{29} = 1.4138\ldots$$

etc.

We obtain a sequence of fractions that get closer and closer to $\sqrt{2}$.

$$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots$$

However, did you know that $\sqrt{2}$ itself can <u>not</u> be written as a fraction?

Theorem (Pythagoreans, c. 500 BC):

$\sqrt{2}$ can not be written as a fraction.
(we say it is "irrational")

Proof: Assume (hoping for a contradiction) that we can write

$$\sqrt{2} = a/b \text{ with } a, b \in \mathbb{N}.$$

We might as well assume that a, b have no common factor. Now square both sides to get

$$2 = a^2/b^2$$
$$2b^2 = a^2.$$

This implies that a^2 is even, and hence a is even (by Quiz 4.5). Let's say $a = 2k$ for $k \in \mathbb{N}$. Substituting gives

$$2b^2 = a^2 = (2k)^2 = 4k^2$$
$$b^2 = 2k^2$$

This implies that b^2 is even, and hence b is even, say $b = 2l$.

Thus $a = 2k$ and $b = 2l$.

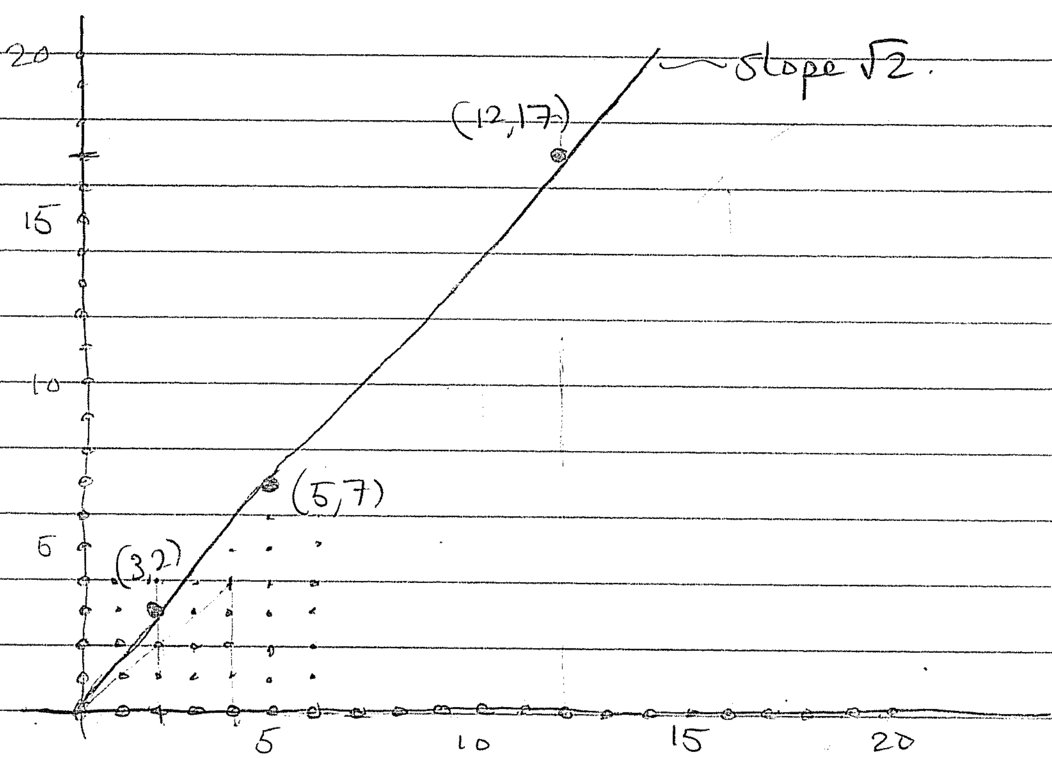
$$\lightning$$

But this contradicts the fact that a and b have no common factor.

This contradiction means that our assumption (that $\sqrt{2}$ can be written as a fraction) is false.

//

There is a more geometric way to say this. Consider a line of slope $\sqrt{2}$ in the Cartesian plane.



The theorem above says this line will never pass through an integer point (a,b) with $a,b \in \mathbb{Z}^2$ (except for $(0,0)$). Of course the line will come arbitrarily close to integer points. The closest points are

$(2,3), (5,7), (12,17), (29,41)$, etc.

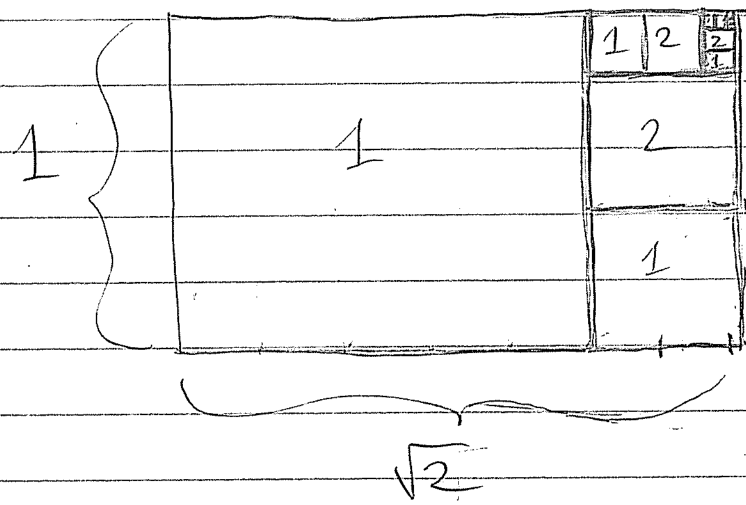
This is just an interpretation of the continued fraction.

Here's another interpretation of the continued fraction.

- Start with a rectangle of dimensions $1 \times \sqrt{2}$.
- Cut off as many squares as you can.
- Repeat.

The number of squares at each step tells you the continued fraction for $\sqrt{2}$.

Picture:



We get the sequence $[1; 2, 2, 2, 2, \ldots]$

This is called the Euclidean Algorithm and it works for any number.

Thinking Problems:

- Why is this algorithm the same as the continued fraction?

- Compute the continued fraction of $\sqrt{3} = 1.7320508\cdots$

The Babylonians also had a cute algorithm for computing $\sqrt{2}$.

Last time I tried and failed to remember why their method is the same as the Euclidean Algorithm

$$\boxed{x_{n+1} = 1 + \frac{1}{1+x_n}}$$

The reason I failed is because their method is slightly different from the Euclidean algorithm.

☆ The Babylonian Algorithm (very old)

- Take a guess at $\sqrt{2}$. Say your guess is x. (It doesn't have to be a good guess.)

- Improve your guess by computing the average of x and $2/x$. This is your new guess.

- Repeat.

Let's try it.

- Our first guess is $x_0 = 1$
- Then we recursively compute

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

$$= \frac{1}{2}\left(\frac{x_n^2+2}{x_n}\right) = \frac{x_n^2+2}{2x_n}$$

So we have

$$x_1 = \frac{1^2+2}{2\cdot 1} = \frac{3}{2} = 1.5$$

$$x_2 = \frac{\left(\frac{3}{2}\right)^2+2}{2\cdot\frac{3}{2}} = \frac{\frac{9}{4}+2}{3} = \frac{17/4}{3} = \frac{17}{12} = 1.41666\cdots$$

$$x_3 = \frac{\left(\frac{17}{12}\right)^2+2}{2\cdot\frac{17}{12}} = \frac{\frac{289}{144}+2}{17/6} = \frac{577/144}{17/6}$$

$$= \frac{577}{144}\cdot\frac{6}{17} = \frac{3462}{2448} = \frac{577}{408}$$

$$= 1.414\overline{2156862745098039}\ldots$$

Recall $\sqrt{2} = 1.414214\ldots$

Wow, the Babylonian Algorithm for $\sqrt{2}$ is <u>much</u> <u>faster</u> than the Euclidean Algorithm for $\sqrt{2}$

In fact, somebody proved that the k^{th} iteration of the B.A. is the same as the $(2^k - 1)^{th}$ iteration of the E.A.

The general Babylonian Algorithm computes $\sqrt{c}$ for any $c > 0$

$$x_0 = 1 \quad (\text{any } x_0 > 0 \text{ will work})$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

In retrospect, we recognize this as an example of

"Newton's Method"

Newton's Method:

It is a sad fact that most equations can not be solved explicitly. For example, it is not possible to express the solution of

$$x^5 - x - 1 = 0$$

in terms of the algebraic operations $+, -, \times, \div, \sqrt{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots$

[This was proved by Abel and Galois in the 1820s.]

So what can we do? Our only hope is to find a method to compute arbitrarily good approximations to the solution.

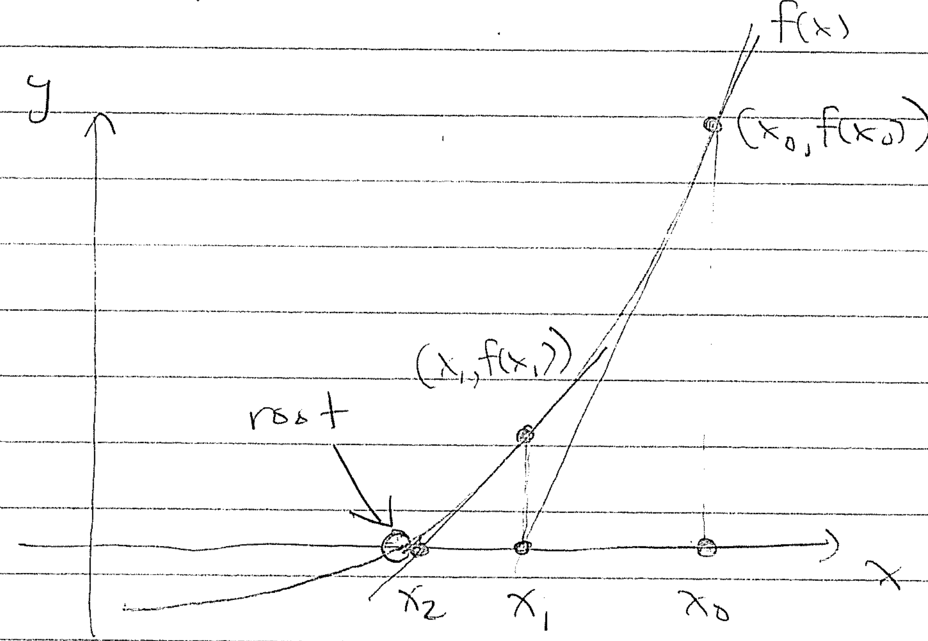
Newton gave a powerful method to do this.

Here's Newton's method:

To solve the equation $f(x) = 0$,

- Make a guess x_0.
- To improve your guess, find the tangent line to the curve $y = f(x)$ at $(x_0, f(x_0))$. Your new guess is the point where this line intersects the x-axis.
- Repeat.

Here's the picture:



The guesses $x_0, x_1, x_2, \ldots$ converge to a root.

Explicitly, the tangent line at $(x_n, f(x_n))$ has slope $f'(x_n)$ and equation

$$\frac{(y - f(x_n))}{(x - x_n)} = f'(x_n).$$

Let $y = 0$ and solve for x.

$$\frac{-f(x_n)}{x - x_n} = f'(x_n)$$

$$x - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is your new guess

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This algorithm works really well if $f(x)/f'(x)$ has a simple form.

For example: To compute $\sqrt{c}$ we use the function $f(x) = x^2 - c$. Then $\sqrt{c}$ is the solution to $f(x) = 0$.

Compute $f'(x) = 2x$, so

$$x_{n+1} = x_n - \frac{(x_n^2 - c)}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 - c}{2x_n}$$

$$= \frac{x_n^2 - c}{2x_n}$$

$$= \frac{1}{2}\left(x_n - \frac{c}{x_n}\right).$$

In this case Newton's method is just the Babylonian Algorithm.

Thinking Problem:

Use Newton's method to approximate the 3rd root of 2, $\sqrt[3]{2}$.

Newton's method is one of the most important algorithms in mathematics.

It has analogues in higher dimensions and Kantorovich (1940) proved that it converges very quickly!

8/1/14

HW 5 due now
Quiz 5 Monday
etc.

Last time we saw two methods to compute $\sqrt{2}$. We called them the

Euclidean Algorithm,

$$x_0 = 1$$
$$x_{n+1} = 1 + \frac{1}{1+x_n}$$

and the Babylonian Algorithm

$$x_0 = 1$$
$$x_{n+1} = \frac{x_n^2 - 2}{2x_n}.$$

We saw that the B.A. leads to Newton's Method which has wide applications.

The Euclidean Algorithm also has wide applications, but it leads in a <u>different</u> direction, to Gaussian Elimination in linear algebra.

Here's an example:

Suppose you live in a country where the coins come in two denominations,

¢12 coins
¢31 coins

For which amounts of money can you make change?

We want to solve the equation

$$12x + 31y = d$$

where x, y, d are non-negative integers.
(This is called a "Diophantine equation".)

To simplify matters we will first let x, y, d be possibly negative integers

We can solve this problem with the Euclidean Algorithm.

In a nutshell, the E.A. says:

- Given two integers $a, b \in \mathbb{N}$.
- Subtract the smaller from the larger. Replace the larger number with the result.
- Repeat until you get 0.

Try with the pair $(a,b) = (12, 31)$

$$
\begin{aligned}
(12,31) &\longrightarrow 31-12 = 19 \\
(12,19) &\longrightarrow 19-12 = 7 \\
(7,12) &\longrightarrow 12-7 = 5 \\
(5,7) &\longrightarrow 7-5 = 2 \\
(2,5) &\longrightarrow 5-2 = 3 \\
(2,3) &\longrightarrow 3-2 = 1 \\
(1,2) &\longrightarrow 2-1 = 1 \\
(1,1) &\longrightarrow 1-1 = 0 \quad \text{DONE.}
\end{aligned}
$$

Another way to say this is by writing $31/12$ as a "continued fraction".

$$\frac{31}{12} = 1 + \frac{19}{12} = 2 + \frac{7}{12}$$

$$= 2 + \frac{1}{12/7}$$

$$= 2 + \frac{1}{1 + 5/7}$$

$$= 2 + \frac{1}{1 + \frac{1}{7/5}}$$

$$= 2 + \frac{1}{1 + \frac{1}{1 + 2/5}}$$

$$= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5/2}}}$$

$$= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}}}$$

We have found that

$$\frac{31}{12} = [2; 1, 1, 2, 2].$$

So what? How does this help us solve

$$12x + 31y = z \ ?$$

For this we use something called the "extended Euclidean algorithm".

Write down very easy solutions and then combine them to get more.

There are two easy solutions:

$$12 \cdot 0 + 31 \cdot 1 = 31$$
$$12 \cdot 1 + 31 \cdot 0 = 12$$

We can subtract the second from the first to get

$$12 \cdot (0-1) + 31 (1-0) = 31 - 12$$
$$12 \cdot (-1) + 31 (1) = 19$$

Still not surprising, but we can continue to combine solutions:

$$\begin{array}{rl} & 12(-1) + 31(1) = 19 \\ - & 12(1) + 31(0) = 12 \\ \hline & 12(-2) + 31(1) = 7. \end{array}$$

Keep going. To save space let's write this in tabular form.

x	y	z	
0	1	31	(1)
1	0	12	(2)
-2	1	7	(3) = (1) − 2·(2).
3	-1	5	(4) = (2) − (3).
-5	2	2	(5) = (3) − (4)
13	-5	1	(6) = (4) − 2·(5)
-31	12	0	(7) = (5) − 2·(6) DONE.

The final two rows are more interesting. They say

$$12(13) + 31(-5) = 1$$
$$12(-31) + 31(12) = 0$$

In fact, the <u>complete</u> solution is contained in these two equations.

Note that we can multiply each by a constant

$$12(13d) + 31(-5d) = d$$
$$12(-31k) + 31(12k) = 0$$

Finally, we add these to get

$$\boxed{12(13d - 31k) + 31(12k - 5d) = d}$$

Claim: As d, k range over $\mathbb{Z}$, this gives <u>all</u> of the integer solutions.

Proof: omitted. ///

In summary:

The general solution to $12x + 31y = z$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13d - 31k \\ 12k - 5d \\ d \end{pmatrix} \quad \forall\, d, k \in \mathbb{Z}.$$

Great! So, which solutions are non-negative?

We need $x = 13d - 31k \geq 0$

and $y = 12k - 5d \geq 0$.

See what happens:

$$13d - 31k \geq 0$$
$$13d \geq 31k.$$
$$\frac{13d}{31} \geq k$$

and
$$12k - 5d \geq 0$$
$$12k \geq 5d.$$
$$k \geq \frac{5d}{12}.$$

Hence we require

$$\frac{13d}{31} \geq k \geq \frac{5d}{12}$$

We're very close now.

Consider any non-negative amount of money $d \in \mathbb{N}$.

We've shown that we can make change for ¢d using coin values ¢12 and ¢31 <u>if and only if</u> $\exists\, k \in \mathbb{Z}$ such that

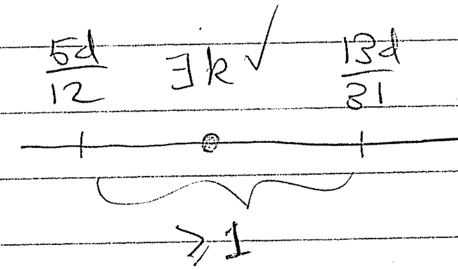
$$\frac{5d}{12} \leq k \leq \frac{13d}{31}.$$

For each such k we get a solution

$$12\underset{\substack{\uparrow \\ \text{This many} \\ \text{¢12 coins}}}{\left(13d - 31k\right)} + 31\underset{\substack{\uparrow \\ \text{This many} \\ \text{¢31 coins.}}}{\left(-5d + 12k\right)} = d.$$

So for which amounts d does a solution exist? Well there will certainly be an integer between $\frac{5d}{12}$ and $\frac{13d}{31}$ if

$$\frac{13d}{31} - \frac{5d}{12} \geq 1$$



We can solve this:

$$\frac{13d}{31} - \frac{5d}{12} \geq 1.$$

$$\left(\frac{13}{31} - \frac{5}{12}\right) d \geq 1.$$

$$\left(\frac{13(12) + 31(-5)}{31 \cdot 12}\right) d \geq 1.$$

$$\left(\frac{1}{372}\right) d \geq 1$$

$$d \geq 372.$$

Conclusion: We can make change for any value $d \geq$ ¢372. Below that the problem is trickier.

Natural Q: What is the largest d that can not be represented by ¢12 and ¢31 coins?

The Answer (The Frobenius Coin Problem):

Consider $a, b \in \mathbb{N}$ with no common factor (we say a, b are "coprime"). If your country has ¢ a and ¢ b coins then

- All but finitely many values can be represented
- The largest non-representable value is

$$¢(ab - a - b).$$

- There are exactly $(a-1)(b-1)/2$ non-representable values. They are a little tricky to describe.

Here are the non-representable values for $(a, b) = (5, 8)$:

27	22	17	12	7	2
19	14	9	4		
11	6	1			
3					

Do you see how this diagram was made?