Math 309 **Summer 2014**
Quiz 4 Solutions **Drew Armstrong**

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1. Accurately state the Well-Ordering Axiom for $\mathbb{N}$.

Every nonempty subset of $\mathbb{N}$ has a smallest element.

2. Accurately state the Principle of Induction for $\mathbb{N}$.

Consider a function $P : \mathbb{N} \to \{T, F\}$. If

- $P(b) = T$ for some $b \in \mathbb{N}$, and
- for all $k \geq b$ we have $P(k) \Rightarrow P(k+1)$,

then it follows that $P(n) = T$ for all $n \geq b$.

Recall that we define the Boolean function $\Leftrightarrow$ by

$$\text{“}P \Leftrightarrow Q\text{”} := \text{“}P \Rightarrow Q\text{”} \wedge \text{“}Q \Rightarrow P\text{”}.$$

3. Draw the truth table for $P \Leftrightarrow Q$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

4. Express $P \Leftrightarrow Q$ in terms of the basic operations $\vee, \wedge, \neg$.

The disjunctive normal form is

$$P \Leftrightarrow Q = (P \wedge Q) \vee (\neg P \wedge \neg Q).$$

We could also use the fact that $P \Rightarrow Q = \neg P \vee Q$ and $Q \Rightarrow P = \neg Q \vee P$ to write

$$P \Leftrightarrow Q = (\neg P \vee Q) \wedge (\neg Q \vee P).$$

5. Consider $n \in \mathbb{Z}$. Prove that the following statement is true by proving its contrapositive: “If n^2 is even, then n is even.”

Proof. Let $P =$“n^2 is even” and $Q =$“n is even”. We wish to prove that $P \Rightarrow Q$. To do this we will prove the equivalent statement $\neg Q \Rightarrow \neg P$. In other words, “if n is odd then n^2 is odd”. So **assume** that n is odd, say $n = 2k+1$ for some $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 2k + 2k + 1 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Since $n^2 = 2(\text{some integer}) + 1$ we conclude that n^2 is odd. $\square$