Math 309 **Summer 2014**
Homework 3 Solutions **Drew Armstrong**

Let k and n be integers such that $0 \leq k \leq n$. Then we define:

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}$$

1. Use algebra to verify that for relevant values of k and n we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. We find a common denominator and then add the fractions to get

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-k)}{(n-k)} \cdot \frac{(n-1)!}{k!(n-k-1)!} + \frac{k}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-k)(n-1)!}{k!(n-k)!} + \frac{k(n-1)!}{k!(n-k)!} \\
&= \frac{[(n-k)+k](n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

□

2. Now give a **counting argument** for the identity in Problem 1. [Hint: Consider the set of binary strings of length n containing k 1's. Divide these into two kinds of strings: those with leftmost symbol 0 and those with leftmost symbol 1. How many of each kind are there?]

Proof. Recall that $\binom{n}{k}$ is the number of binary strings of length n containing k 1's. Let S be the set of such strings. We can decompose S as a union of two disjoint subsets

$$S = S_0 \sqcup S_1,$$

where S_0 is the set of binary strings of length n containing k 1's, whose leftmost symbol is "0", and S_1 is the set whose leftmost symbol is "1". We know (for example, from our recent discussion of probability) that

$$\binom{n}{k} = \#S = \#S_0 + \#S_1.$$

Now I claim that $\#S_0 = \binom{n-1}{k}$ and $\#S_1 = \binom{n-1}{k-1}$. Indeed, if the leftmost symbol is 0 then the remaining symbols form a binary string of length $n-1$ containing k 1's, and there are $\binom{n-1}{k}$

of these. And if the leftmost symbol is 1 then the remaining $n-1$ symbols form a binary string of length $n-1$ containing $k-1$ 1's (because the leftmost symbol is one of the k 1's), and there are $\binom{n-1}{k-1}$ of these. We conclude that

$$\binom{n}{k} = \#S = \#S_0 + \#S_1 = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

□

[An example will be helpful. Here are the binary strings of length 5 containing 3 1's.

11100	01110
11010	01101
11001	01011
10110	00111
10101	
10011	

Note that there are $\binom{5}{3} = 10$ of these and we can divide them into two groups. The strings on the left have first symbol 1. The number of these is $\binom{4}{2} = 6$ because by deleting the first symbol we obtain the strings of length 4 with 2 1's:

1100 1010 1001 0110 0101 0011.

The strings on the right have first symbol 0. The number of these is $\binom{4}{3} = 4$ because by deleting the first symbol we obtain the strings of length 4 with 3 1's:

1110 1101 1011 0111.

This explains **why** $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$.]

It seems that the notation $\binom{n}{k}$ only makes sense when k and n are integers such that $0 \le k \le n$. However, note that we can rewrite the formula as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{(n)_k}{k!},$$

where $(n)_k := n(n-1)(n-2)\cdots(n-k+1)$. The interesting thing about this is that $(z)_k$ makes sense for **any real or complex number** z. This allows us to define

$$\binom{z}{k} := \frac{(z)_k}{k!},$$

where $k \in \mathbb{N}$ and z is any real or complex number. Why would we want to do this?

3. Use the definition $\binom{z}{k} := (z)_k/k!$ to evaluate the following.

(a) If $k, n \in \mathbb{N}$ with $k > n$, show that $\binom{n}{k} = 0$.

(b) For $k \in \mathbb{N}$ and any integer n, show that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Proof. For part (a), suppose that $k, n \in \mathbb{N}$ with $k > n$. Let's say that $k = n + d$ with $d \geq 1$. By definition we have $\binom{n}{k} = (n)_k/k!$ where

$$\begin{aligned}(n)_k &= n(n-1)(n-2)\cdots(n-k+1)\\ &= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot 0\cdot(-1)\cdot(-2)\cdot\cdots\cdot(-d+1)\\ &= 0.\end{aligned}$$

The product is zero because it contains 0 as a factor. We conclude that $\binom{n}{k} = 0/k! = 0$. This still seems reasonable because, for example, the number of ways to choose 7 different things from a set of 5 is zero.

For part (b), let k and n be integers such that $k \geq 0$ (so $k!$ is defined). Then we have

$$\begin{aligned}\binom{-n}{k} &= \frac{(-n)_k}{k!}\\ &= \frac{-n(-n-1)(-n-2)\cdots(-n-k+1)}{k!}\\ &= \frac{(-1)n(-1)(n+1)(-1)(n+2)\cdots(-1)(n+k-1)}{k!}\\ &= \frac{(-1)^k(n+1)(n+2)\cdots(n+k-1)}{k!}\\ &= \frac{(-1)^k[n+k-1][(n+k-1)-1]\cdots[(n+k-1)-k+2][(n+k-1)-k+1]}{k!}\\ &= \frac{(-1)^k(n+k-1)_k}{k!}\\ &= (-1)^k\binom{n+k-1}{k!}.\end{aligned}$$

In fact, I notice that we didn't even need n to be an integer. This equation is true for any complex number n. □

[Is the equation from part (b) reasonable? Does it have anything to do with counting? Maybe ... but a more obvious application comes from Isaac Newton. He showed that the numbers $\binom{z}{k}$ where z is any complex number show up as the coefficients of the Taylor series for the function $(1+x)^z$ near $x = 0$.]

4. Let x and z be any complex numbers with $|x| < 1$. Isaac Newton proved that

$$(1+x)^z = 1 + \binom{z}{1}x + \binom{z}{2}x^2 + \binom{z}{3}x^3 + \cdots,$$

where the right hand side is a convergent infinite series.

(a) Show that this gives the usual Binomial Theorem when $z := n \in \mathbb{N}$.
(b) Use Newton's formula to obtain an infinite series expansion of $(1+x)^{-2}$.

Proof. For part (a), let $n \in \mathbb{N}$. Then from our result in Problem 3(a), Newton's formula says

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n + 0 + 0 + 0 + \cdots,$$

which is just the usual Binomial Theorem.

For part (b) we first use our formula from Problem 3(b) to see that

$$\binom{-2}{k} = (-1)^k \binom{2+k-1}{k} = (-1)^k \binom{k+1}{k} = (-1)^k (k+1).$$

Then Newton's formula says

$$\begin{aligned}\frac{1}{(1+x)^2} &= 1 + \binom{-2}{1}x + \binom{-2}{2}x^2 + \binom{-2}{3}x^3 + \cdots \\ &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots .\end{aligned}$$

□

[Wow, that power series from Problem 4(b) looks like the derivative of something. It **is** the derivative of something. Recall the famous "geometric series":

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

This is true for x near 0. Substituting $-x$ for x gives

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots ,$$

which is still true for x near 0. Note that this is just Newton's binomial theorem with $z = -1$. Finally, if we differentiate both sides of this series we get

$$\begin{aligned}\frac{-1}{(1+x)^2} &= 0 - 1 + 2x - 3x^2 + 4x^3 - \cdots \\ \frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 - 4x^3 + \cdots .\end{aligned}$$

What happens if you differentiate the series again? Maybe you will start to see how Newton discovered his binomial theorem.]