Math 309 **Summer 2014**
Homework 3 **Drew Armstrong**

Let k and n be integers such that $0 \leq k \leq n$. Then we define:

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}$$

1. Use algebra to verify that for relevant values of k and n we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

2. Now give a **counting argument** for the identity in Problem 1. [Hint: Consider the set of binary strings of length n containing k 1's. Divide these into two kinds of strings: those with leftmost symbol 0 and those with leftmost symbol 1. How many of each kind are there?]

It seems that the notation $\binom{n}{k}$ only makes sense when k and n are integers such that $0 \leq k \leq n$. However, note that we can rewrite the formula as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{(n)_k}{k!},$$

where $(n)_k := n(n-1)(n-2)\cdots(n-k+1)$. The interesting thing about this is that $(z)_k$ makes sense for **any real or complex number** z. This allows us to define

$$\binom{z}{k} := \frac{(z)_k}{k!},$$

where $k \in \mathbb{N}$ and z is any real or complex number. Why would we want to do this?

3. Use the definition $\binom{z}{k} := (z)_k/k!$ to evaluate the following.

(a) If $k, n \in \mathbb{N}$ with $k > n$, show that $\binom{n}{k} = 0$.

(b) For $k \in \mathbb{N}$ and any integer n, show that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

4. Let x and z be any complex numbers with $|x| < 1$. Isaac Newton proved that

$$(1+x)^z = 1 + \binom{z}{1}x + \binom{z}{2}x^2 + \binom{z}{3}x^3 + \cdots,$$

where the right hand side is a convergent infinite series.

(a) Show that this gives the usual Binomial Theorem when $z := n \in \mathbb{N}$.

(b) Use Newton's formula to obtain an infinite series expansion of $(1+x)^{-2}$.