Math 309 **Exam 1**
Fall 2022 **Tues Sept 27**

No electronic devices are allowed. There are 5 page and 5 problems. Each problem is worth 6 points, for a total of 30 points.

1. Binomial Coefficients.

(a) Draw Pascal's Triangle down to the sixth row.

$$
\begin{array}{ccccccccccccc}
&&&&&&1&&&&&&\\
&&&&&1&&1&&&&&\\
&&&&1&&2&&1&&&&\\
&&&1&&3&&3&&1&&&\\
&&1&&4&&6&&4&&1&&\\
&1&&5&&10&&10&&5&&1&\\
1&&6&&15&&20&&15&&6&&1
\end{array}
$$

(b) Use part (a) to expand the polynomial $(1+x)^6$.

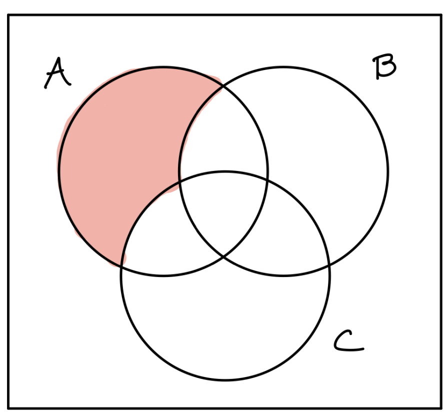
$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

(c) Use part (a) to evaluate the following sum:

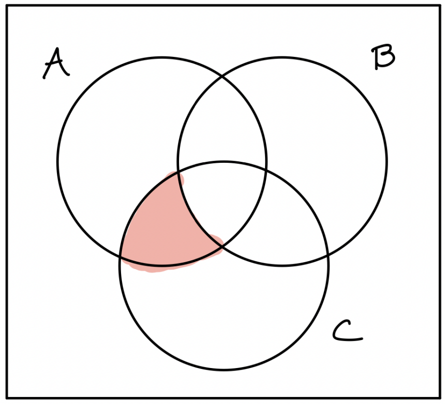
$$
\begin{aligned}
&\sum_{k=0}^{4}(3-k)\binom{6}{k}\\
&= (3-0)\binom{6}{0} + (3-1)\binom{6}{1} + (3-2)\binom{6}{2} + (3-3)\binom{6}{3} + (3-4)\binom{6}{4}\\
&= 3\cdot 1 + 2\cdot 6 + 1\cdot 15 + 0\cdot 20 - 1\cdot 15\\
&= 15
\end{aligned}
$$

2. Venn Diagrams. Let A, B, C be subsets of the universal set U.

(a) Draw a Venn diagram to illustrate the set $A \cap B' \cap C'$.



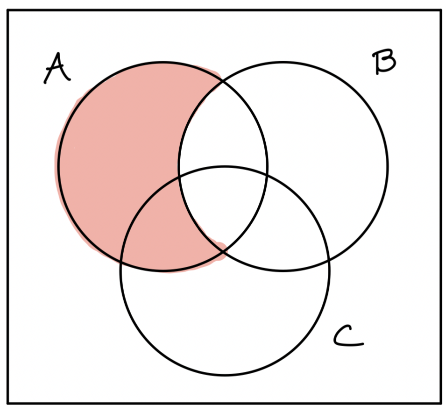
(b) Draw a Venn diagram to illustrate the set $A \cap B' \cap C$.



(c) Use your diagrams from (a) and (b) to find a simpler expression for the set

$$(A \cap B' \cap C') \cup (A \cap B' \cap C).$$

This is the union of the sets from parts (a) and (b):



From the picture we see that this set is equal to $A \cap B'$.

3. Truth Tables.

(a) Complete the following truth table.

P	Q	$P \vee Q$	$P \wedge Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

(b) Use a truth table to show that $(P \wedge Q) \Rightarrow P$ is true for any values of P and Q.

P	Q	$P \wedge Q$	$(P \wedge Q) \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

(c) Consider the Boolean function $f(P, Q, R)$ defined by the following truth table:

P	Q	R	$f(P, Q, R)$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

Write an expression for $f(P, Q, R)$ using the basic operations $\vee, \wedge, \neg$.

(Infinitely many correct answers.) There are Ts in the rows corresponding to $P \wedge \neg Q \wedge R$ and $P \wedge \neg Q \wedge \neg R$, hence the disjunctive normal form is

$$f(P, Q, R) = (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R).$$

Observe that this is the same expression as in Problem 2(c). Hence we also have

$$f(P, Q, R) = P \wedge \neg Q.$$

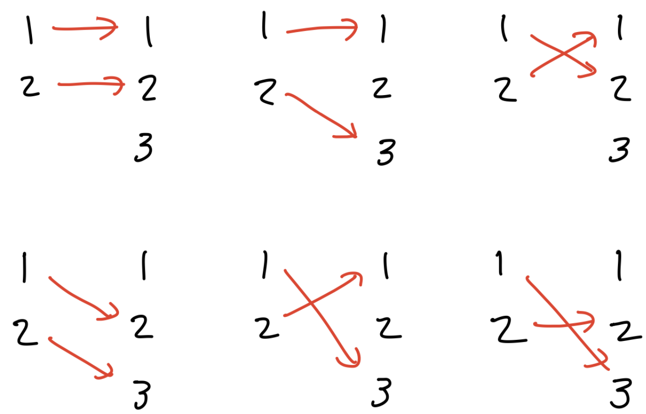
4. Counting Functions. Compute the number of each kind of function.

(a) All functions from $\{1, 2\}$ to $\{1, 2, 3\}$.

The number of functions $S \to T$ between finite sets is $\#T^{\#S}$. In our case we have $\#S = 2$ and $\#T = 3$, so the number of possible functions is $3^2 = 9$.

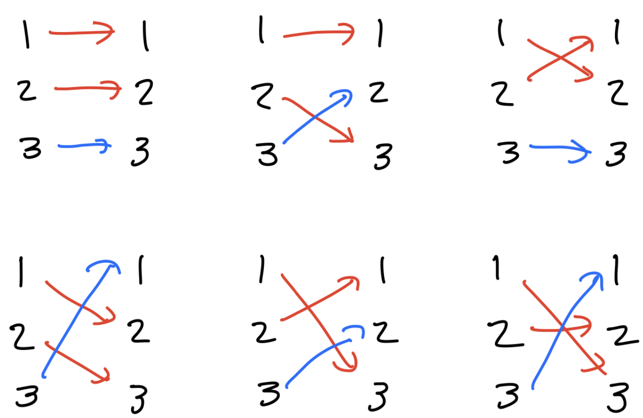
(b) Injective functions from $\{1, 2\}$ to $\{1, 2, 3\}$. [Hint: Draw them.]

In general, the number of injective functions from a set of size k to a set of size n is $n(n-1)\cdots(n-k+1)$, since there are n ways to choose where the first element goes, then $n-1$ ways to choose where the second element goes, etc. In our case we have $k = 2$ and $n = 3$ so the number of injective functions is $3 \cdot 2 = 6$. Picture:



(c) Injective functions from $\{1,2,3\}$ to $\{1,2,3\}$. [Hint: Draw them.]

This time we have $k = 3$ and $n = 3$, so the number of injective functions is $3 \cdot 2 \cdot 1 = 6$. Here is a picture:



Remarks.

- An injective function between two sets of the same size must also be surjective, and hence bijective. The number of bijective functions between two sets of size n is $n(n-1)\cdots 3 \cdot 2 \cdot 1 = n!$. Such functions are also called *permutations*.
- Do you notice any similarity between the pictures in 4(b) and 4(c)?

5. Induction. Define a sequence $c_0, c_1, c_2, \ldots$ by the following recurrence:

$$c_n = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \\ 5c_{n-1} - 6c_{n-2} & n \geq 2. \end{cases}$$

(a) Compute the first few terms of the sequence and fill in the following table:

n	0	1	2	3	4
c_n	1	2	4	8	16

(b) Try to guess a formula for the nth term of the sequence.

I guess that $c_n = 2^n$.

(c) Use induction to prove that your formula from part (b) is correct. [Hint: You will need two base cases.]

Proof by Strong Induction. Consider the statement $P(n) =$ " $c_n = 2^n$ ". Since c_n is defined by a second order recurrence, we need to check two base cases. Indeed, we observe that $P(0)$ and $P(1)$ are true. Now fix some arbitrary integer $n \geq 1$ and assume for induction that $P(k) = T$ for all $0 \leq k \leq n$.[1] That is, we assume that $c_k = 2^k$ for all $0 \leq k \leq n$. It follows that

$$\begin{aligned} c_{n+1} &= 5c_n - 6c_{n-1} && \text{by definition} \\ &= 5 \cdot 2^n - 6 \cdot 2^{n-1} && \text{by } P(n) \text{ and } P(n-1) \\ &= 10 \cdot 2^{n-1} - 6 \cdot 2^{n-1} \\ &= (10 - 6) \cdot 2^{n-1} \\ &= 4 \cdot 2^{n-1} \\ &= 2^{n+1}, \end{aligned}$$

and hence $P(n+1)$ is also true. □

Remark. Where did I come up with this problem? The theory of second order recurrences tells us that the recurrence $c_n = 5c_{n-1} - 6c_{n-2}$ has general solution $c_n = a \cdot 2^n + b \cdot 3^n$ for some constants a and b, because 2 and 3 are the roots of the equation $x^2 = 5x - 6$. The constants are determined by the initial conditions. In this case I chose the initial conditions so that $a = 1$ and $b = 0$. This is a nice way to get a tricky-looking recurrence with a simple-looking solution.

[1] Actually we will only use $P(n)$ and $P(n-1)$.