10/13/14

I am out of town right now,
Someone named Brittney Ellzey is talking to you. She might say what is written here; she might say something different.

Boolean Algebra is done!
The new topic is the Binomial Theorem.

Last week we discussed how the subsets of

$$U = \{1, 2, 3, \ldots, n\}$$

can be encoded as binary strings.
The subset $A \subseteq U$ corresponds to string

$$b_1 b_2 b_3 \cdots b_n$$

where the ith "bit" is $b_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$

Example: Here are the subsets of $\{1,2,3\}$ and their corresponding binary strings

| | |
|---|---|
| $\{1,2,3\}$ | 111 |
| $\{1,2\}$ $\{1,3\}$ $\{2,3\}$ | 110 101 011 |
| $\{1\}$ $\{2\}$ $\{3\}$ | 100 010 001 |
| $\varnothing$ | 000. |

This gives us a convenient way to count subsets.

Theorem: Let $U$ be a set with $n$ elements. Then $U$ has exactly $2^n$ different subsets.

Proof: This is the same as counting binary strings of length $n$. There are 2 choices for each bit and there are $n$ independent choices to make.

Thus the total number of choices is

$$\underset{1st}{\underline{2}} \times \underset{2nd}{\underline{2}} \times \underset{3rd}{\underline{2}} \times \ldots \times \underset{nth}{\underline{2}} = 2^n$$

This week we are interested in a more refined problem.

<u>Problem</u>: Let $U$ be a set with $n$ elements and consider an integer $k$ such that $0 \leq k \leq n$. Then <u>how many</u> subsets with $k$ elements does $U$ have?

Equivalently, how many of the $2^n$ binary strings of length $n$ contain exactly $k$ 1's (and hence $n-k$ 0's)?

<u>Example</u>

111 —— 1

110 101 011 —— 3

100 010 001 —— 3

000 —— 1.

We are interested in this equation:

$$2^3 = 1 + 3 + 3 + 1 .$$

$$2^n = \text{what ?}$$

---

To solve this we will need a "preliminary fact" (which we call a "lemma").

Q: Given $n$ different symbols, in how many ways can I write them in a line?

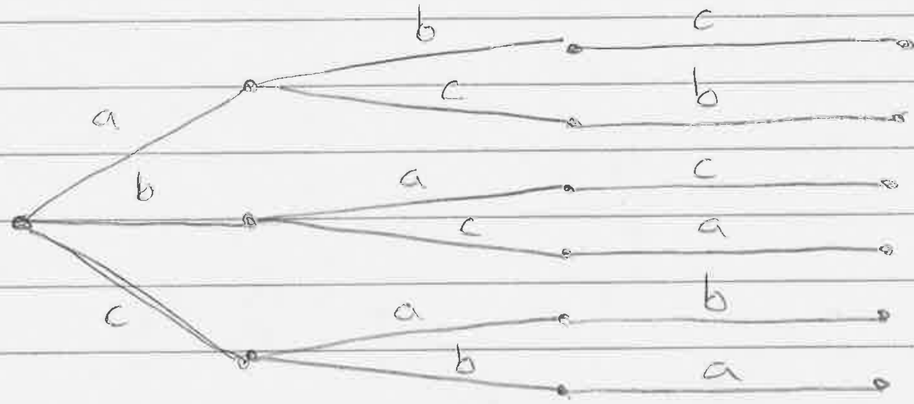
Example: Using symbols $a, b, c$ the possibilities are:

abc, acb, bac, bca, cab, cba.

We call these the permutations of the symbols. There are 6 of them.

Q: How many permutation are there of $n$ different symbols?

First let's note that $6 = 3 \times 2 \times 1$. We can arrange the permutations of $a, b, c$ in a tree like this:




So we really want to count the branches of this tree. The total # of branches is

$$\underset{1st}{\underline{3}} \times \underset{2nd}{\underline{2}} \times \underset{3rd}{\underline{1}} = 6$$

In general, given a positive integer $n$ we define the notation

$$n! := n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

We call this "n factorial".

Lemma: The number of permutations of $n$ different symbols is $n!$

Proof: There are $n$ ways to choose the first/leftmost symbol. Then there are $n-1$ remaining choices for the 2nd symbol. Continuing in this way, the total number of choices is

$$\underbrace{n}_{1st} \times \underbrace{n-1}_{2nd} \times \underbrace{n-2}_{3rd} \times \cdots \times \underbrace{2}_{(n-1)^{th}} \times \underbrace{1}_{n^{th}} = n!$$

//

These numbers grow fast!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
| $n!$ | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | ... |

James Stirling (1692–1770) gave a charming and surprising formula for their rate of growth. He proved

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

"Stirling's Approximation".

That is surprising right?

---

However, the problem of permutations is not so easy when some of the symbols are the same.

Example: How many permutations of

$$a, a, b, b \quad ?$$

aabb, abab, abba, baab, baba, bbaa

Answer: It's not $4! = 24$. It's just 6.

Example: How many permutations of

$$a, a, a, b, b, b, b \quad ?$$

Now it's too many to do by hand. We need to think about it systematically somehow...

We need a trick.

Here's the trick: Let's temporarily label the symbols.

$$a_1, a_2, a_3, b_1, b_2, b_3, b_4.$$

Now we know that there are

$$7! = 5040$$

permutations. But this number is too big because many of these correspond to the same un<u>labeled</u> permutations.

For example, the <u>labeled</u> permutations

$$a_3 b_4 a_1 b_1 b_3 a_2 b_2 \ \& \ a_1 b_3 a_3 b_1 b_2 a_2 b_4$$

both correspond to the un<u>labeled</u> permutation

$$ababbab.$$

We need to find out exactly how often this happens.

For example, how many times does ababbab show up in our count of 5040?

Actually this not too hard. There are $3! = 6$ ways to label the a's and $4! = 24$ ways to label the b's.

$$\text{ababbab} \to \begin{array}{c} a_1 b_1 a_2 b_2 b_3 a_3 b_4 \\ a_1 b_1 a_3 b_2 b_3 a_2 b_4 \\ \text{etc.} \end{array}$$

$$3! \times 4! = 6 \times 24 = 144.$$

In fact, every unlabeled permutation will get counted 144 times because because there is always 144 ways to label it.

Now we can solve the problem.

Let $N$ = The number of unlabeled permutations of a, a, a, b, b, b, b. Since each of these got counted 144 times, we conclude that

$$7! = N \times 3! \times 4!$$
$$5040 = 144 \cdot N$$

$$\Longrightarrow N = \frac{5040}{144} = 35.$$

(I guess we could have counted those by hand, but it would have taken a while, and we probably would have made mistakes.)

---

Now we will use the same trick to solve the general problem.

<u>Theorem</u>: Let $U$ be a set with $n$ elements and let $k$ be an integer such that $0 \leq k \leq n$.

Then the number of subsets of $\mathcal{U}$ with $k$ elements is given by

$$\frac{n!}{k!\,(n-k)!}$$

Proof: This is the same as counting permutations of the symbols

$$\underbrace{1, 1, 1, \ldots, 1}_{k \text{ of these}}, \underbrace{0, 0, 0, \ldots, 0}_{n-k \text{ of these}}$$

i.e., binary strings of length $n$ containing $k$ 1's and $n-k$ 0's.
Let $N$ be the number of such strings.
We want to find an equation for $N$.

To do this we will consider an auxiliary problem, to count the permutations of the labeled symbols

$$1_1, 1_2, 1_3, \ldots, 1_k, 0_1, 0_2, \ldots, 0_{n-k}$$

On one hand there are $n!$ such permutations because these $n$ symbols are all different.

On the other hand, these <u>labeled</u> permutations break up into groups corresponding to the different un<u>labeled</u> permutations. Each of these groups has the same size $k!(n-k)!$, because given any un<u>labeled</u> permutation there are $k!(n-k)!$ ways to label it. [ $k!$ ways to label the $1$'s and $(n-k)!$ ways to label the $0$'s. ]

We conclude that

$$n! = N \times \underbrace{k! \times (n-k)!}_{\text{label them}}$$

order the labeled symbols

order the unlabeled symbols

Hence

$$N = \frac{n!}{k!(n-k)!}$$

Some Remarks:

- It is not obvious that $n!/(k!(n-k)!)$ is even an <u>integer</u>, but we just proved that it is, because it <u>counts</u> something.

- The method we used is called "double counting": Count a certain set in two different ways to get an equation. It is very useful.

- Wait! Is our formula true when $k=0$ or $k=n$?

Hmm...

It depends what you mean by $0!$

10/15/14

NO CLASS ON MONDAY OCT 20
HW4 due Mon Oct 27
Exam 2 on Wed Oct 29.

Current Topic: Binomial Theorem.

Last time you proved the following.

☆ Theorem: Let $U$ be a set with $n$ elements and let $k$ be an integer such that $0 \le k \le n$. The number of subsets of $U$ of size $k$ is equal to

$$\frac{n!}{k!(n-k)!}$$

We will use a special notation for these numbers

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

"n choose k".

Example: I have 10 (different) books on my shelf and I want to give you 4 of them. In how many ways can I do this?

$$\binom{10}{4} = \frac{10!}{4!\,6!}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7\cdot \cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{4\cdot 3\cdot 2\cdot 1\cdot \cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}$$

$$= \frac{10\cdot \cancelto{3}{9}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 10\cdot 3\cdot 7 = 210.$$

///

Discussion:

- We observe that $\binom{n}{k} = \binom{n}{n-k}$. There are two ways to see this.

i) Directly from the formula:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

///

ii) Using a counting argument. Let $\# U = n$. Let $X \subseteq \wp(U)$ be the set of subsets containing $k$ elements and let $Y \subseteq \wp(U)$ be the set of subsets containing $n-k$ elements. Then we have a bijection

$$\begin{array}{ccc} X & \longleftrightarrow & Y \\ A & \longmapsto & A^c \\ B^c & \longleftarrow\!\!\!\shortmid & B \end{array}$$

By HW2.1c we conclude that

$$\binom{n}{k} = \# X = \# Y = \binom{n}{n-k}.$$

Example: $U = \{1, 2, 3\}$, $k = 1$.

$$\begin{array}{ccc} \{1\} & \longleftrightarrow & \{2,3\} \\ \{2\} & \longleftrightarrow & \{1,3\} \\ \{3\} & \longleftrightarrow & \{1,2\} \end{array}$$

$$\binom{3}{1} = 3 \qquad \binom{3}{2} = 3.$$

• We know that $\binom{n}{0} = \binom{n}{n} = 1$, but does this agree with the formula?

$$\binom{n}{0} = \frac{\cancel{n!}}{0!\,\cancel{n!}} = \frac{1}{0!} = ?$$

Wait! What is $0!$?

$$0! = 0 \cdot (-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot \dots$$

That makes no sense. So we will just <u>define</u> it to be

$$0! := 1.$$

Ok? Then

$$\binom{n}{0} = \binom{n}{n} = \frac{1}{0!} = 1$$

as desired. This is mostly a notational convenience, but there are deeper reasons (involving the "gamma function"

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad ).$$

- Since the total number of subsets of $U$ is $2^n$ we have a nice equation

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Examples:

$$1 = 1 \qquad \left(2^0 = \binom{0}{0}\right)$$
$$2 = 1 + 1$$
$$4 = 1 + 2 + 1$$
$$8 = 1 + 3 + 3 + 1$$
$$16 = 1 + 4 + 6 + 4 + 1.$$

Actually, this is just the shadow of a more interesting equation. Let $a, b$ be <u>any</u> numbers and consider the number

$$(a+b)^n.$$

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Do you recognize this?

The official statement is called

☆ The Binomial Theorem: For all numbers $a$ & $b$ and for all integers $n \geq 0$ we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

How can we prove this?

Proof: Let's temporarily pretend that $ab \neq ba$.

Then we have, for example,

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a(a+b) + b(a+b) \\ &= aa + ab + ba + bb \\ &= aa + \begin{pmatrix} ab \\ + \\ ba \end{pmatrix} + bb\end{aligned}$$

$$(1 \quad 2 \quad 1)$$

and $(a+b)^3 = (a+b)(a+b)^2$
$= (a+b)(aa + ab + ba + bb)$

$= aaa + aab + aba + abb$
$+ baa + bab + bba + bbb$

$$= aaa + \begin{pmatrix} aab \\ + \\ aba \\ + \\ baa \end{pmatrix} + \begin{pmatrix} abb \\ + \\ bab \\ + \\ bba \end{pmatrix} + bbb.$$

$$( \underline{1} \qquad 3 \qquad 3 \qquad 1 ).$$

In general, we see that $(a+b)^n$ is the sum of all words of length $n$ using the letters $a$ & $b$. We know that the number of such words containing $k$ $a$'s and $n-k$ $b$'s equals $\binom{n}{k}$. [How do we know this?]

Thus if we allow $ab = ba$ then the term $a^k b^{n-k}$ will occur $\binom{n}{k}$ times in the exponsion of $(a+b)^n$. In other words,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

///

[Recall how we counted the words with $k$ $a$'s and $n-k$ $b$'s. Let $N$ be the number of such words. We will count permutations of the symbols

$$a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_{n-k}$$

in two different ways:

$$\underset{\substack{\uparrow \\ \#\text{ labeled} \\ \text{words}}}{n!} = \underset{\substack{\uparrow \\ \#\text{ unlabeled} \\ \text{words}}}{N} \cdot \underbrace{k! \cdot (n-k)!}_{\substack{\uparrow \\ \#\text{ ways to} \\ \text{Label them.}}}$$

]

Q: What good is the Binomial Theorem?

A: Well, it remains true for any values of $a$ & $b$ that we substitute.

For example, if we put $a=1$ and $b=-1$ then we get

$\Downarrow$

$$(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^k (-1)^{n-k}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}.$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

\# even subsets = \# odd subsets

[Wait! What about when $n = 0$ ? ]

We can even treat $(a+b)^n$ as a function of a & b and do things like differentiate it.

Example: Let $a = x$ and $b = 1$, so

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n.$$

Now differentiate both sides by $x$:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}$$

Now substitute $x=1$ to get

$$n \cdot 2^{n-1} = 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$

That was pure algebra. On the other hand this equation has an interpretation in terms of counting subsets.

<u>Problem</u>: Let $U$ be a set of $n$ people. Find the number of ways of choosing a committee with a president.

One one hand we can choose the president <u>first</u> in $n$ ways and then we can choose the other committee members in $2^{n-1}$ ways, for a total of

$$\underset{\substack{\uparrow \\ \text{choose} \\ \text{president}}}{n} \cdot \underset{\substack{\uparrow \\ \text{choose the other } n-1 \\ \text{committee members.}}}{2^{n-1}} \quad \text{choices.}$$

On the other hand, we could choose the committee first and then the president. If the committe has $k$ members then the number of choices is

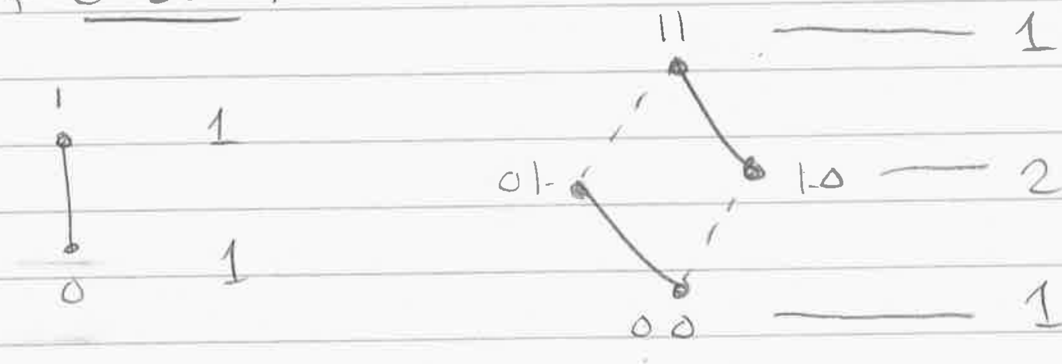
$$\underbrace{\binom{n}{k}}_{\text{choose the committee}} \cdot \underbrace{k}_{\text{then choose the president from the committee.}}$$

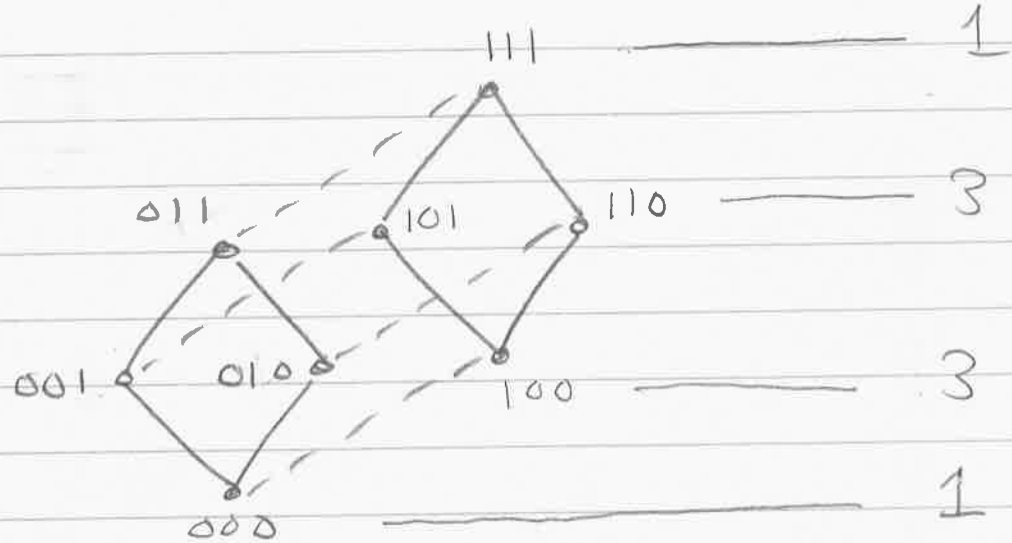
To get the total number of choices we sum over all possible sizes of committee:

$$1 \cdot \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}.$$

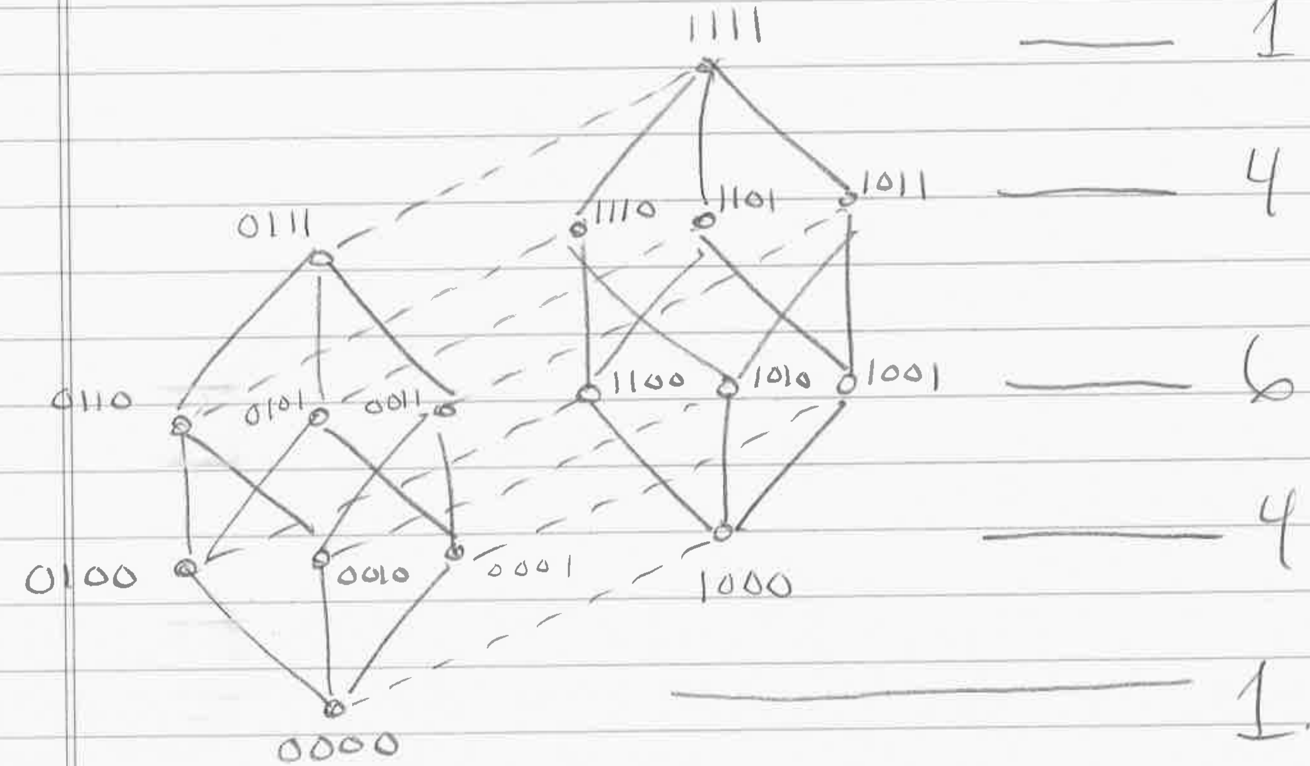
Which proof do you like better: the counting proof or the algebra?

Next wednesday we will discuss the recursive structure of the Binomial Theorem. It has to do with the recursive structure of <u>cubes</u>.



4 dimensional cube:




and so on.

10/22/14

HW 4 due next Mon
Exam 2 on Wed.
We'll do review and discuss HW 4 on Mon.

Today: Pascal's Triangle.

We have seen that the numbers $\binom{n}{k}$ (read "n choose k") have several interpretations.

- $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
- $\binom{n}{k}$ = The number of ways to choose $k$ unordered things from a set of $n$ unordered things.
- $\binom{n}{k}$ = The number of binary strings with $k$ 1's and $n-k$ 0's.
- The Binomial Theorem:
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

The fact that these four interpretations are equivalent needs to be proved, which we proved in the previous two classes. Once we know this fact we can apply it.

<u>Example</u>: How many "words" can you make using all of the letters

$$a, a, a, b, b, b, b \quad ?$$

Answer: $\binom{7}{3} = \binom{7}{4} = \frac{7!}{3!4!}$

$$= \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

[Do you remember how to prove this? Count the words you can make from

$$a_1, a_2, a_3, b_1, b_2, b_3, b_4$$

in two different ways:

$$7! = X \cdot 3! \cdot 4!$$
]

Example: How many subsets of size 3 does the set $\{1,2,3,4,5,6,7\}$ have?

Answer: 35 again, because its the same problem. There is a bijection between the words and the subsets.

$$\{2,4,5\} \subseteq \{1,2,\dots,7\} \iff \underset{(1\,2\,3\,4\,5\,6\,7)}{babaabb}.$$

Example: Expand $(a+b)^7$ by hand.

Answer:

$$(a+b)^7 = \binom{7}{0}b^7 + \binom{7}{1}ab^6 + \binom{7}{2}a^2b^5 + \binom{7}{3}a^3b$$

$$+ \binom{7}{4}a^4b^3 + \binom{7}{5}a^5b^2 + \binom{7}{6}a^6b + \binom{7}{7}a^7.$$

But it could take a few minutes to compute these coefficients. I'll show you a helpful shortcut.

They are just the entries in the 7th row of "Pascal's Triangle"

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

So the answer is

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$$
$$+ 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

Why does that work? Let's figure out first exactly what it is that is working. We claim that the $k^{th}$ entry of the $n^{th}$ row of Pascal's Triangle equals

$$\binom{n}{k}.$$

That is,

$$\begin{array}{c}
\binom{0}{0} \\[6pt]
\binom{1}{0} \quad \binom{1}{1} \\[6pt]
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\[6pt]
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}
\end{array}$$

etc.

Pascal's Triangle is <u>defined</u> by the fact that each entry is the sum of the two above.

$$\begin{array}{ccc}
\binom{n-1}{k-1} & & \binom{n-1}{k} \\
& \searrow \quad \swarrow & \\
& \binom{n}{k} &
\end{array}$$

So we need to prove that for all relevant values of $n$ and $k$ we have

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.}$$

We can use *any* of the different interpretations to prove this. You'll give two different proofs on HW 4. Here is a third proof, using the Binomial Theorem.

☆ Theorem: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

*Proof*: We will take for granted the fact that for all numbers $a, b$ and for all integers $n \geq 0$ we have

$$(a+b)^n = \binom{n}{0} b^n + \binom{n}{1} a b^{n-1} + \cdots + \binom{n}{n-1} a^{n-1} b + \binom{n}{n} a^n.$$

Then we will use a *very small trick*:

$$\begin{aligned}(a+b)^n &= (a+b)(a+b)^{n-1} \\ &= a(a+b)^{n-1} + b(a+b)^{n-1}.\end{aligned}$$

Now we just put everything together:

(It won't fit here. Turn the page.)

$$\binom{n}{0} b^n + \binom{n}{1} a b^{n-1} + \cdots + \binom{n}{k} a^k b^{n-k} + \cdots + \binom{n}{n} b^n.$$

$$= a\left[\binom{n-1}{0} b^{n-1} + \cdots + \binom{n-1}{n-1} a^{n-1}\right]$$

$$+ b\left[\binom{n-1}{0} b^{n-1} + \cdots + \binom{n-1}{n-1} a^{n-1}\right]$$

$$= \left[0 + \binom{n-1}{0} a b^{n-1} + \binom{n-1}{1} a^2 b^{n-1} + \cdots + \binom{n-1}{n-2} a^{n-1} b + \binom{n-1}{n-1} a^n\right]$$

$$+ \left[\binom{n-1}{0} b^n + \binom{n-1}{1} a b^{n-1} + \binom{n-1}{2} a^2 b^{n-1} + \cdots + \binom{n-1}{n-1} a^{n-1} b + 0\right]$$

$$= \left(0 + \binom{n-1}{0}\right) b^n + \left(\binom{n-1}{0} + \binom{n-1}{1}\right) a b^{n-1} + \cdots$$

$$\cdots + \left(\binom{n-1}{k-1} + \binom{n-1}{k}\right) a^k b^{n-k} + \cdots$$

$$\cdots + \left(\binom{n-1}{n-2} + \binom{n-1}{n-1}\right) a^{n-1} b + \left(\binom{n-1}{n-1} + 0\right) a^n.$$

Comparing the coefficient of $a^k b^{n-k}$ on both sides gives

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

//

Remark: For the equation to work at the ends we need to say that

$$\binom{n}{-1} = \binom{n}{n+1} = 0.$$

We <u>will</u> say this. In fact, for all $n \geq 0$ we will say that

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n. \\ 0 & \text{otherwise} \end{cases}$$

This shows that the binomial coefficients $\binom{n}{k}$ are the same as the entries of P.T.

$$\begin{array}{c}
\cdots\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \cdots \\
\cdots\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \cdots \\
\cdots\ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \cdots \\
\cdots\ 0\ \ 0\ \ 0\ \ 1\ \ 3\ \ 3\ \ 1\ \ 0\ \ 0\ \ 0\ \cdots \\
\cdots\ 0\ \ 0\ \ 1\ \ 4\ \ 6\ \ 4\ \ 1\ \ 0\ \ 0\ \cdots \\
\cdots\ 0\ \ 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1\ \ 0\ \cdots \\
\cdots\ 1\ \ 6\ \ 15\ \ 20\ \ 15\ \ 6\ \ 1\ \cdots
\end{array}$$

I could have phrased this in a different way. I could have asked you to solve the following recurrence and initial conditions:

$$\boxed{\begin{array}{l} \bullet \; f(0,k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases} \\[2em] \bullet \; f(n,k) = f(n-1,k) + f(n-1,k-1) \quad \forall n,k \in \mathbb{Z}, n \geq 1. \end{array}}$$

Wow, this is more complicated than previous recusion problems, but we already know the answer.

Theorem: The solution is $f(n,k) = \binom{n}{k}$.

Proof by induction on $n$:

(1) First we verify the base case. Indeed, we know that

$$\binom{0}{k} = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

hence $f(0,k) = \binom{0}{k}$ for all $k \in \mathbb{Z}$.

(2) Now fix some $n \geq 1$ and assume for induction that we have

$$f(n,k) = \binom{n}{k} \quad \forall\, k \in \mathbb{Z}.$$

In this hypothetical case we want to prove that

$$f(n+1,k) = \binom{n+1}{k} \quad \forall\, k \in \mathbb{Z}.$$

Indeed, for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} f(n+1,k) &= f(n,k) + f(n,k-1) && \text{by definition} \\ &= \binom{n}{k} + \binom{n}{k-1} && \text{by induction hypothesis} \\ &= \binom{n+1}{k} && \text{by the Theorem proved earlier in today's class.} \end{aligned}$$

By induction we conclude that we have

$$f(n,k) = \binom{n}{k} \quad \forall\, k \in \mathbb{Z}.$$

for <u>all</u> $n \geq 0$. $\blacksquare$

Remark: I don't expect you to solve general recurrences that complicated. This is a very special case.

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On the HW4 you will give two more proofs that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

One of them is a counting argument. Consider the set of binary strings with $k$ 1's and $n-k$ 0's. Divide them into two sets based on their leftmost bit.

Example: $n=5$, $k=3$.

| | |
|---|---|
| 11100 | 01110 |
| 11010 | 01101 |
| 11001 | 01011 |
| 10110 | 00111 |
| 10101 | |
| 10011 | |

$$6 + 4 = 10$$

What do you see?

10/27/14

HW 4 due NOW

Exam 2 on Wed.

Today: Discuss HW 4 and review for Exam 2.

Topics for Exam 2:

- "Abstract" Boolean Algebra
  - proving formulas using algebraic manipulation instead of Venn diagrams or truth tables.
    (e.g. using de Morgan's Law)

- Properties of the Boolean functions

$$\oplus, \Rightarrow, \Leftrightarrow, \uparrow,$$

- Basic Notions of counting
  - Let A, B be finite sets. The size of the Cartesian product is

$$\#(A \times B) = \#A \times \#B.$$

- The number of functions from $A$ to $B$ is

$$\#B^{\#A}$$

Example: The number of Boolean functions in $n$ variables,

$$\varphi : \{T,F\}^n \to \{T,F\},$$

is $\#\{T,F\}^{\#(\{T,F\}^n)} = \#\{T,F\}^{(\#\{T,F\})^n}$

$$= 2^{2^n}.$$

- Subsets of $U$ = Functions $U \to \{T,F\}$.

There is a natural bijection between subsets of $U$ and functions $U \to \{T,F\}$ given by sending the subset $A \subseteq U$ to the function $f_A : U \to \{T,F\}$ defined by

$$f_A(x) = \begin{cases} T & x \in A \\ F & x \notin A \end{cases}$$

The inverse sends the function $f: U \to \{T, F\}$ to the subset

$$\{x \in U : f(x) = T\}.$$

We conclude that the number of subsets of $U$ equals the number of functions $U \to \{T, F\}$, i.e.,

$$\#\{T,F\}^{\#U} = 2^{\#U}.$$

- Subsets = Binary Strings.

We can also encode a subset $A \subseteq U$ as a binary string with $\#A$ "1"s and $\#U - \#A$ "0"s.

Example

$$\{2,6,7\} \subseteq \{1,\ldots,7\} \leftrightarrow 0100011$$

- Counting Subsets

Let $\#\mathcal{U} = n$. The total # of subsets of $\mathcal{U}$ is $2^n$, but how many subsets of each size?

Let $\binom{n}{k}$ = # subsets of size $k$.

☆ **Theorem:** $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Proof: Instead we count binary strings of length $n$ with $k$ "1"s. By counting the permutations of symbols

$$1_1, 1_2, \ldots, 1_k, 0_1, 0_2, \ldots, 0_{n-k}$$

in two different ways, we find that

$$n! = \binom{n}{k} \cdot k!(n-k)!$$

Hence

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

//

- The Binomial Theorem
  - says that for all numbers a & b and all integers $n \geq 0$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

- Pascal's Triangle.
  - is defined by the following recurrence

$$\boxed{\begin{array}{l} \bullet\ f(0,k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases} \\[2ex] \circ\ f(n,k) = f(n-1,k) + f(n-1,k-1) \quad \forall\, n,k \in \mathbb{Z},\ n \geq 1 \end{array}}$$

$k = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots$

| $n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n=0$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | |
| 2 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | | |
| 3 | 0 | 1 | 3 | 3 | 1 | 0 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | |

etc.

Theorem : $f(n,k) = \binom{n}{k}$.

This is proved in two steps

(1) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

(2) Use induction to show that

$$f(n,k) = \binom{n}{k}.$$

Discussion of HW 4 :

A standard deck of cards contains 26 red and 26 black cards.

A "hand" of cards consists of 5 cards.

The number of possible hands is

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot \overset{10}{\cancel{50}} \cdot 49 \cdot \overset{2}{\cancel{48}} \cdot 47!}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot 47!}$$

$$= 52 \cdot 51 \cdot 10 \cdot 49 \cdot 2 = 2{,}598{,}960.$$

The number of hands with 2 red and 3 black cards is

$$\binom{26}{2}\binom{26}{8} = \frac{26!}{2!\,24!}\;\frac{26!}{3!\,23!}$$

choose 2 red cards　　choose 2 black cards

$$= \frac{\cancel{26}^{13}\cdot 25}{\cancel{2}} \cdot \frac{\cancel{26}^{13}\cdot 25\cdot \cancel{24}^{8}}{\cancel{3}\cdot\cancel{2}}$$

$$= 13\cdot 25\cdot 13\cdot 25\cdot 8$$

$$= 845{,}000.$$

In general, the number of hands with $k$ red cards and $5-k$ black cards is

$$\binom{26}{k}\binom{26}{5-k}$$

choose $k$ red cards　　choose $n-k$ black cards.

Since every hand has <u>some</u> number of red cards, we get

$$\binom{52}{5} = \sum_k \binom{26}{k}\binom{26}{5-k}.$$

$$= \binom{26}{0}\binom{26}{5} + \binom{26}{1}\binom{26}{4} + \binom{26}{2}\binom{26}{3}$$

$$+ \binom{26}{3}\binom{26}{2} + \binom{26}{4}\binom{26}{1} + \binom{26}{5}\binom{26}{0}$$

More generally, suppose we have a deck of cards with $R$ red cards and $B$ black cards, and suppose a "hand" consists of $n$ cards.

The total # of possible hands is

$$\binom{R+B}{n}.$$

The number of hands with $k$ red cards is

$$\underbrace{\binom{R}{k}}_{\text{choose } k \text{ red cards}}\underbrace{\binom{B}{n-k}}_{\text{choose } n-k \text{ black cards}}$$

Since every hand has some number of red cards we get

$$\binom{R+B}{n} = \underbrace{\sum_k \binom{R}{k}\binom{B}{n-k}}_{\text{some terms may be zero if } R<n \text{ or } B<n.}$$

Example: $R=2$, $B=4$, $n=4$.

$$\binom{2+4}{4} = \binom{2}{0}\binom{4}{4} + \binom{2}{1}\binom{4}{3} + \binom{2}{2}\binom{4}{2} + \binom{2}{3}\binom{4}{1} + \binom{2}{4}\binom{4}{0}$$

$$15 = 1\cdot 1 + 2\cdot 4 + 1\cdot 6 + 0\cdot 4 + 0\cdot 1$$

$$15 = 1 + 8 + 6 + 0 + 0 \quad \checkmark$$

Later we will interpret this in terms of probability: Suppose you are dealt 4 cards from a deck with 2 red and 4 black cards.

What is the probability that you get exactly one red card?

P( one red card)

$$= \frac{\text{\# ways to get one red card}}{\text{total \# possible hands}}$$

$$= \frac{\binom{2}{1}\binom{4}{3}}{\binom{6}{4}} = \frac{2 \cdot 4}{15} = \frac{8}{15} = 0.5333\ldots$$

The probability of getting exactly one red card is 53.3 %