Math 309 **Fall 2014**
Homework 4 **Drew Armstrong**

Let k and n be integers such that $0 \leq k \leq n$. In this case we define the notation

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}.$$

By convention we will say that $\binom{n}{k} := 0$ if $k < 0$ or $k > n$.

1. Consider a standard deck of 52 cards. A subset of 5 cards is called a "hand".

(a) How many different hands are there?
(b) How many different hands are there with all red cards?
(c) How many different hands are there that contain 2 red and 3 black cards?

2. Vandermonde Convolution. Suppose you have an urn containing R red balls and G green balls. You reach into the urn and grab n balls. Use this situation to give a counting argument for the following identity:

$$\sum_k \binom{R}{k}\binom{G}{n-k} = \binom{R+G}{n}.$$

3. Use the formula to verify that for relevant values of k and n we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

4. Give a **counting argument** for the identity in Problem 3. [Hint: Consider the set of binary strings of length n containing k "1"s. How many are there? Now break this set into two subsets: the strings with leftmost symbol "0" and the strings with leftmost symbol "1". How many are there of each kind?]

5. Trinomial Coefficients. Consider integers $i, j, k \geq 0$ such that $i + j + k = n$. Let N be the number of different words of length n containing i "a"s, j "b"s, and k "c"s. Explain why

$$n! = N \cdot i! \cdot j! \cdot k!$$

[Hint: Count the permutations of the symbols $a_1, \ldots, a_i, b_1, \ldots, b_k, c_1, \ldots, c_j$ in two different ways.] Use the result to compute the number of different words (not necessarily English words) that can be made from the letters

$$b, a, n, a, n, a.$$