Math 309 **Fall 2014**
Homework 1 Solutions **Drew Armstrong**

For all positive integers p and n we consider the sum of the first n p-th powers:

$$S_p(n) := 1^p + 2^p + 3^p + \cdots + n^p = \sum_{k=1}^{n} k^p.$$

We have already seen that $S_1(n) = n(n+1)/2$ and $S_2(n) = n(n+1)(2n+1)/6$. Now you will show that $S_3(n) = n^2(n+1)^2/4$ using the technique of induction.

1. Verify that $S_3(n) = n^2(n+1)^2/4$ is true for small values of n.

When $n = 1$ the equation

$$1^3 = S_3(1) = \frac{1^2 \cdot 2^2}{4} = 1$$

is true. When $n = 2$ the equation

$$1^3 + 2^3 = S_3(2) = \frac{2^2 \cdot 3^2}{4} = 9$$

is true. When $n = 3$ the equation

$$1^3 + 2^3 + 3^3 = S_3(3) = \frac{3^2 \cdot 4^2}{4} = 36$$

is true. Technically, for the proof we only need to check the case $n = 1$. I did the other two just for fun, and to make sure that I actually believe the formula. In real life I would probably have my computer check a whole bunch of cases.

2. Let n be some "fixed, but arbitrary" positive integer. Show that **if** $S_3(n) = n^2(n+1)^2/4$ is true **then** $S_3(n+1) = (n+1)^2(n+2)^2/4$ is also true. [Hint: Take out common factors whenever you can.]

[Warning: Look carefully at the words I use in the proof. They are not random. If your proof doesn't have any words at all, then it is not correct. At a bare minimum, your proof must contain the words "if ... then".]

Let n be some **fixed but arbitrary** positive integer. We will **assume for induction** that the equation

$$S_3(n) = \frac{n^2(n+1)^2}{4}.$$

is true. **In this hypothetical case**, we want to show that the equation

$$S_3(n+1) = \frac{((n+1))^2((n+1)+1)^2}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

is **also** true. To do this we begin by considering **the definition** of $S_3(n+1)$. Recall that

$$S_3(n+1) = 1^3 + 2^3 + 3^3 + \cdots + (n+1)^3.$$

How can we prove anything about this? We only know about $S_3(n)$. Aha! Let's try to express $S_3(n+1)$ (which we **don't** know) in terms of $S_3(n)$ (which we **do** know). Good idea. Now let's finish the proof. We have

$$\begin{aligned} S_3(n+1) &= 1^3 + 2^3 + \cdots + (n+1)^3 \\ &= (1^3 + 2^3 + \cdots + n^3) + (n+1)^3 \\ &= S_3(n) + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2}{4} + (n+1)\right) \\ &= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\ &= \frac{(n+1)^2(n+2)^2}{4}, \end{aligned}$$

as desired. The proof is done.

[Remark: Hey, did you notice that $S_3(n) = S_1(n)^2$? That's weird. Why would that happen?]

3. Use your knowledge of $S_1(n)$, $S_2(n)$, and $S_3(n)$ to find a closed form for the following sum:

$$\sum_{k=1}^{n} k(k+1)(k+2).$$

First we expand the summand:

$$k(k+1)(k+2) = k(k^2+3k+2) = k^3 + 3k^2 + 2k.$$

Now we distribute the sum:

$$\begin{aligned} \sum_{k=1}^{n} k(k+1)(k+2) &= \sum_{k=1}^{n}(k^3 + 3k^2 + 2k) \\ &= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\ &= S_3(n) + 3\cdot S_2(n) + 2\cdot S_1(n) \\ &= \frac{n^2(n+1)^2}{4} + 3\cdot\frac{n(n+1)(2n+1)}{6} + 2\cdot\frac{n(n+1)}{2} \\ &= n(n+1)\left(\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right) \\ &= n(n+1)\left(\frac{n(n+1) + 2(2n+1) + 4}{4}\right) \\ &= n(n+1)\left(\frac{n^2 + n + 5n + 2 + 4}{4}\right) \\ &= \frac{n(n+1)(n+2)(n+3)}{4}. \end{aligned}$$

[Remark: Hey, it's pretty cool that it the formula factors like that. Why does that happen? We'll see a good reason later.]

For the next two problems, consider the recurrence relation:

$$\boxed{F_n = F_{n-1} + 2n.}$$

4. (a) Compile a table of F_n with initial condition $F_0 = 0$.
(b) Compile a table of F_n with initial condition $F_2 = 5$.
(c) If $F_7 = x$ then what is F_3?

With initial condition $F_0 = 0$ we have

n	0	1	2	3	4	5	6	7	$\cdots$
F_n	0	2	6	12	20	30	42	56	$\cdots$

With initial condition $F_2 = 5$ we have

n	0	1	2	3	4	5	6	7	$\cdots$
F_n	-1	1	5	11	19	29	40	55	$\cdots$

Note that to get from the first table to the second we just subtract 1 from each F_n. I'll bet I can use that observation to solve part (c). In the first table we have $F_7 = 56$. To get from 56 to x we should add $x - 56$. In the first table we have $F_3 = 12$. Adding $x - 56$ to this gives $12 + (x - 56) = x - 44$. So I guess that $F_3 = x - 44$. Is there a more rigorous way to do this? Yes. We can rewrite the recurrence as $F_{n-1} = F_n - 2n$. If $F_7 = x$ then we have

$$\begin{aligned}
F_6 &= F_7 - 14 = x - 14 \\
F_5 &= F_6 - 12 = x - 14 - 12 = x - 26 \\
F_4 &= F_5 - 10 = x - 26 - 10 = x - 36 \\
F_3 &= F_4 - 8 = x - 36 - 8 = x - 44.
\end{aligned}$$

5. Find a **closed formula** for F_n with initial condition $F_0 = 0$. [Hint: Expand the recurrence to show that $F_n = 0 + 2 + 4 + \cdots + 2n$. Now what?]

Using the initial condition $F_0 = 0$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= F_0 + 2 = 0 + 2 \\
F_2 &= F_1 + 4 = 0 + 2 + 4 \\
&\vdots \\
F_n &= 0 + 2 + 4 + \cdots + 2n.
\end{aligned}$$

We can rewrite this and use the formula $S_1(n) = n(n+1)/2$ to get

$$F_n = 2 + 4 + 6 + \cdots + 2n = 2(1 + 2 + 3 + \cdots + n) = 2 \cdot \frac{n(n+1)}{2} = n(n+1).$$

That formula is about as "closed" as you can get. Good work.