Math 309 | **Fall 2014**
Exam 2 (Wed Oct 29) | **Drew Armstrong**

There are 5 problems, with a total of 20 parts. Each part is worth 2 points, for a total of 40 points. If two exams are submitted with identical answers then **both** will receive 0 points.

1. Boolean Algebra. Recall that the Boolean function $\Rightarrow$ is defined by

$$P \Rightarrow Q := (\neg P) \vee Q.$$

(a) Draw the truth table for $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(b) Accurately state De Morgan's Law.

For all Boolean variables P and Q we have

- $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$
- $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$

(c) Use De Morgan's Law to prove that $(P \Rightarrow Q) = ((\neg Q) \Rightarrow (\neg P))$.

Oops. De Morgan's Law is not actually necessary to prove this. Sorry about that. We have

$$\begin{aligned}(\neg Q) \Rightarrow (\neg P) &= \neg(\neg Q) \vee (\neg P)\\ &= Q \vee (\neg P)\\ &= (\neg P) \vee Q\\ &= P \Rightarrow Q.\end{aligned}$$

(d) Use a truth table to prove that $(P \Rightarrow Q) \neq (Q \Rightarrow P)$.

Note that $P \Rightarrow Q$ and $Q \Rightarrow P$ differ in the second and third rows:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

2. Boolean Functions. A Boolean function with m inputs and n outputs has the form

$$\varphi : \{T, F\}^m \to \{T, F\}^n.$$

(a) Explicitly write down all the elements of the set $\{T,F\}^3$.

Recall that $\{T,F\}^3 = \{T,F\}\times\{T,F\}\times\{T,F\}$ consists of **ordered triples** of elements from $\{T,F\}$. The elements of the set are:

$$\begin{array}{ccc} & (T,T,T) & \\ (T,T,F) & (T,F,T) & (F,T,T) \\ (T,F,F) & (F,T,F) & (F,F,T) \\ & (F,F,F) & \end{array}$$

(b) How many elements does the set $\{T,F\}^n$ have?

$$\#(\{T,F\}^n) = (\#\{T,F\})^n = 2^n$$

(c) How many functions are there from $\{T,F\}^m$ to $\{T,F\}^n$?

The number of functions from $\{T,F,\}^m$ to $\{T,F\}^n$ is

$$\#(\{T,F\}^n)^{\#(\{T,F\}^m)} = (2^n)^{(2^m)}$$

(d) How many Boolean functions are there with 3 inputs and 1 output?

When $m=3$ and $n=1$ the number of functions is

$$(2^n)^{(2^m)} = (2^1)^{(2^3)} = 2^8 = 256.$$

3. Subsets $\leftrightarrow$ Binary Strings. Consider the set $U = \{1,2,3,4,5\}$.

(a) Make a table to display the number of subsets of U with size k, for $k = 0,1,2,3,4,5$.

k	0	1	2	3	4	5
# subsets of size k	1	5	10	10	5	1

(b) Explicitly write down all of the subsets of U containing **two elements**.

$$\begin{array}{llll} \{1,2\}, & \{1,3\}, & \{1,4\}, & \{1,5\}, \\ \{2,3\}, & \{2,4\}, & \{2,5\}, & \\ \{3,4\}, & \{3,5\}, & & \\ \{4,5\} & & & \end{array}$$

(c) Explicitly write down all of the binary strings with two "1"s and three "0"s.

11000, 10100, 10010, 10001,
01100, 01010, 01001,
00110, 00101,
00011

(d) Draw lines between your answers to (b) and (c) to demonstrate a natural bijection.

The bijection is implied by the way I drew the two sets.

4. The Binomial Theorem.

(a) Accurately state the Binomial Theorem.

For any number a and b, and for any integer $n \geq 0$ we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

(b) Use the Binomial Theorem to prove that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.

Subsitute $a = 1$ and $b = 1$.

(c) Use the Binomial Theorem to prove that $3^n = 1\binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \cdots + 2^n\binom{n}{n}$.

Substitute $a = 2$ and $b = 1$.

(d) Use the Binomial Theorem to prove that $0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$.

Substitute $a = -1$ and $b = 1$.

5. Binomial Coefficients.

(a) State the formula for $\binom{n}{k}$.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

(b) Use the formula to prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

$$k\binom{n}{k} = k\frac{n!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k)!} = n\frac{(n-1)!}{(k-1)!\,(n-k)!} = n\binom{n-1}{k-1}$$

For parts (c) and (d), suppose you want to choose a committee of k people from a set of n people. One person on the committee will be called the "president".

(c) Explain why the number of ways to do this is $k\binom{n}{k}$.

If we choose the committee first and then the president, there are $\binom{n}{k}$ ways to choose the committee and then k ways to choose the president from the committee. Hence the total number of choices is $\binom{n}{k}k$.

(d) Explain why that the number of ways to do this is $n\binom{n-1}{k-1}$.

It we choose the president first and then the committee, there are n ways to choose the president and then $\binom{n-1}{k-1}$ ways to choose the other $k-1$ members of the committee from the remaining $n-1$ people. Hence the total number of choices is $n\binom{n-1}{k-1}$.