Math 230 **Spring 2019**
Homework 5 **Drew Armstrong**

Problem 1. Fractions. Consider the following set of ordered pairs:

$$S := \{(a,b) : a,b \in \mathbb{Z}, b \neq 0\}.$$

(a) Prove that the following is an *equivalence relation* on the set S:

$$(a,b) \sim (c,d) \quad \Longleftrightarrow \quad ad = bc.$$

(b) Prove that the following two operations are *well-defined* with respect to $\sim$:

$$(a,b) \times (c,d) := (ac, bd)$$
$$(a,b) + (c,d) := (ad + bc, bd).$$

Problem 2. Modular Arithmetic. Fix an integer $n \geq 1$. We proved in class that the following is an equivalence relation on $\mathbb{Z}$, called *congruence mod* n:

$$a \sim_n b \quad \Longleftrightarrow \quad n|(a-b).$$

Prove that addition and multiplication of integers are *well-defined* with respect to $\sim_n$:

(a) For all $a, b, a', b' \in \mathbb{Z}$ with $a \sim_n a'$ and $b \sim_n b'$, prove that $a + b \sim_n a' + b'$.
(b) For all $a, b, a', b' \in \mathbb{Z}$ with $a \sim_n a'$ and $b \sim_n b'$, prove that $ab \sim_n a'b'$.

Problem 3. Linear Congruence Theorem. Fix the modulus $n = 22$. Since $\gcd(7, 22) = 1$ we know from the Linear Congruence Theorem that the equation $7x \sim_{22} 1$ has a solution.

(a) Use the Euclidean Algorithm to find this solution.
(b) Use your answer from part (a) to solve the following linear congruences:

$$7a \sim_{22} 10,$$
$$7b \sim_{22} 11,$$
$$7c \sim_{22} 12.$$

Problem 4. Fermat's Little Theorem. In this problem you will give an induction proof of Fermat's Little Theorem. You may assume the following statement, which we proved in class. For all $a, b, p \in \mathbb{Z}$ with p prime we have

$$(a+b)^p \sim_p a^p + b^p.$$

Now fix a prime $p \in \mathbb{Z}$ and for any $n \in \mathbb{Z}$ consider the statement $P(n) =$ "$n^p \sim_p n$."

(a) Explain why $P(0)$ and $P(1)$ are true.
(b) If $P(n)$ is true, prove that $P(n+1)$ is also true.
(c) If $P(n)$ is true, prove that $P(-n)$ is also true.