Math 230 **Spring 2019**
Homework 2 **Drew Armstrong**

Problem 1. Let P and Q be any mathematical statements.

(a) Use a truth table to prove **de Morgan's Laws**:

$$\neg(P \vee Q) = (\neg P \wedge \neg Q) \quad \text{and} \quad \neg(P \wedge Q) = (\neg P \vee \neg Q).$$

(b) Use a truth table to verify that $(P \Rightarrow Q) = (\neg P \vee Q)$.

(c) Use part (b) to prove the **contrapositive principle**:

$$(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P).$$

Do **not** use a truth table.

(a) Here is a truth table proving that $\neg(P \vee Q) = \neg P \wedge \neg Q$:

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

And here is a truth table proving that $\neg(P \wedge Q) = \neg P \vee \neg Q$:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

(b) And here is a truth table proving that $(P \Rightarrow Q) = (\neg P) \vee Q$:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(c) *Proof.* From part (b) we know that $(A \Rightarrow B) = \neg A \vee B$ for all statements A and B. Substituting $A = P$ and $B = Q$ gives

$$(P \Rightarrow Q) = \neg P \vee Q,$$

and substituting $A = \neg Q$ and $B = \neg P$ gives

$$(\neg Q \Rightarrow \neg P) = \neg(\neg Q) \vee (\neg P) = Q \vee \neg P.$$

Since $\neg P \vee Q = Q \vee \neg P$ we conclude that $(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P)$. $\square$

Problem 2. Let P, Q, R be any mathematical statements.

(a) Use parts (a) and (b) of Problem 1 to verify that

$$P \Rightarrow (Q \vee R) = (P \wedge \neg Q) \Rightarrow R.$$

Again, do **not** use a truth table. [Hint: You can assume that $\neg(\neg A) = A$ and $A \vee (B \vee C) = (A \vee B) \vee C$ for any statements A, B, C.]

(b) Use the logical principle from part (a) to prove that the following silly statement is true for all integers $m, n \in \mathbb{Z}$:

" If m is odd, then either n is even or mn is odd (or both). "

[Hint: What are the statements P, Q, R in this case?]

(a) *Proof.* Recall from 1(b) that we have $(A \Rightarrow B) = \neg A \vee B$ for all statements A and B. Substituting $A = P$ and $B = Q \vee R$ gives

$$P \Rightarrow (Q \vee R) = \neg P \vee (Q \vee R).$$

On the other hand, substiting $A = (P \wedge \neg Q)$ and $B = R$ and then using de Morgan's law gives

$$\begin{aligned}
(P \wedge \neg Q) \Rightarrow R &= \neg(P \wedge \neg Q) \vee R && \text{1(b)} \\
&= (\neg P \vee \neg\neg Q) \vee R && \text{1(a)} \\
&= (\neg P \vee Q) \vee R && \text{assumption} \\
&= \neg P \vee (Q \vee R) && \text{assumption} \\
&= P \Rightarrow (Q \vee R). && \text{from above}
\end{aligned}$$

□

(b) Let $m, n \in \mathbb{Z}$ and consider the following statements:

$$\begin{aligned}
P &= \text{“}m \text{ is odd,”} \\
Q &= \text{“}n \text{ is even,”} \\
R &= \text{“‘}mn \text{ is odd.”}
\end{aligned}$$

I claim that $P \Rightarrow (Q \vee R)$. *Proof.* By part (a) it is enough to prove that $(P \wedge \neg Q) \Rightarrow R$. In other words, we will prove that

" If m is odd and n is odd, then mn is odd. "

So let us assume that m and n are both odd. By definition this means that there exist integers $k, \ell \in \mathbb{Z}$ such that $m = 2k + 1$ and $n = 2\ell + 1$. But then we have

$$\begin{aligned}
mn &= (2k + 1)(2\ell + 1) \\
&= 4k\ell + 2k + 2\ell + 1 \\
&= 2(2k\ell + k + \ell) + 1 \\
&= 2(\text{some integer}) + 1.
\end{aligned}$$

By definition this means that mn is odd. □

Problem 3. In this problem you will prove that $\sqrt{5}$ is irrational.

(a) There are four different ways that an integer can be "not a multiple of 5." List them.

(b) Use part (a) and the contrapositive to prove for all integers n that

$$(n^2 \text{ is a multiple of } 5) \Rightarrow (n \text{ is a multiple of } 5).$$

[Hint: This will be a case-by-case proof.]

(c) Use part (b) and proof by contradiction to show that $\sqrt{5}$ is not a fraction of whole numbers. [Hint: Try to mimic the proof from class as closely as possible.]

(a) If $n \in \mathbb{Z}$ is not a multiple of 5 then one of the following cases holds:

- $n = 5k + 1$ for some $k \in \mathbb{Z}$,
- $n = 5k + 2$ for some $k \in \mathbb{Z}$,
- $n = 5k + 3$ for some $k \in \mathbb{Z}$,
- $n = 5k + 4$ for some $k \in \mathbb{Z}$.

(b) For all $n \in \mathbb{Z}$ we will prove the statement

$$(n \text{ is not a multiple of } 5) \Rightarrow (n^2 \text{ is not a multiple of } 5).$$

Proof. Assume that n is not a multiple of 5. Then from part (a) there are four cases.

- If $n = 5k + 1$ for some $k \in \mathbb{Z}$ then we have
$$n^2 = (5k+1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1.$$
- If $n = 5k + 2$ for some $k \in \mathbb{Z}$ then we have
$$n^2 = (5k+2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4.$$
- If $n = 5k + 3$ for some $k \in \mathbb{Z}$ then we have
$$n^2 = (5k+3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4.$$
- If $n = 5k + 4$ for some $k \in \mathbb{Z}$ then we have
$$n^2 = (5k+4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1.$$

In any case, we conclude that n^2 is not a multiple of 5. □

(c) Let us assume for contradiction that $\sqrt{5}$ is a fraction of whole numbers. Then we can write $\sqrt{5} = a/b$ for some integers $a, b \in \mathbb{Z}$ in "lowest terms," i.e., where a and b have no common factors except for ± 1. Multiply both sides by b and then square to obtain

$$\begin{aligned} \sqrt{5} &= a/b \\ \sqrt{5} \cdot b &= a \\ 5b^2 &= a^2. \end{aligned}$$

Since a^2 is a multiple of 5, part (b) tells us that a is also a multiple of 5, say $a = 5k$. Then substituting and canceling 5 from both sides gives

$$\begin{aligned} 5b^2 &= a^2 \\ 5b^2 &= (5k)^2 \\ \cancel{5}b^2 &= \cancel{5} \cdot 5k^2 \\ b^2 &= 5k^2. \end{aligned}$$

Then since b^2 is a multiple of 5, part (b) tells us that b is also a multiple of 5, say $b = 5\ell$. But this contradicts the fact that a and b have no common factors. This contradiction shows that $\sqrt{5}$ is **not** a fraction of whole numbers. □

Problem 4. For any integer $n \in \mathbb{Z}$ consider the following mathematical statement:

$$P(n) := \text{“}1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).\text{”}$$

(a) Verify that the statements $P(1)$, $P(2)$ and $P(3)$ are all true.
(b) Now fix an arbitrary positive integer $k \geq 1$ and **assume for induction** that the statement $P(k)$ is true. In this case prove that the statement $P(k+1)$ is also true.
(c) What do you conclude from this?

(a) Note that the following statements are true:

$$\begin{aligned} P(1) &= \text{“}1^2 = \frac{1}{6}(1)(2)(3),\text{''} \\ P(2) &= \text{“}1^2 + 2^2 = \frac{1}{6}(2)(3)(5),\text{''} \\ P(3) &= \text{“}1^2 + 2^2 + 3^2 = \frac{1}{6}(3)(4)(7).\text{''} \end{aligned}$$

(b) Now fix an arbitrary integer $k \geq 1$ and **assume for induction** that $P(k)$ is a true statement. In other words, we assume that

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1).$$

In this case we want to prove that $P(k+1)$ is also true. In other words, we want to prove

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{1}{6}(k+1)((k+1)+2)(2(k+1)+1) = \frac{1}{6}(k+1)(k+2)(2k+3).$$

To see that this is true, observe that

$$\begin{aligned} 1^2 + 2^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + \cdots k^2) + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 && \text{by assumption} \\ &= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right] \\ &= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 1\right] \\ &= \frac{1}{6}(k+1)\left[2k^2 + 7k + 1\right] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3). \end{aligned}$$

□

(c) By the Principle of Induction, we conclude that $P(n)$ is a true statement for **all** $n \geq 1$.

[Remark: Since $1^2 + 2^2 + \cdots + n^2$ is always a whole number, it follows from this result that

$$n(n+1)(2n+1) \text{ is always a multiple of 6.}$$

That's a bit surprising.]