Math 230 Homework 1

Spring 2019
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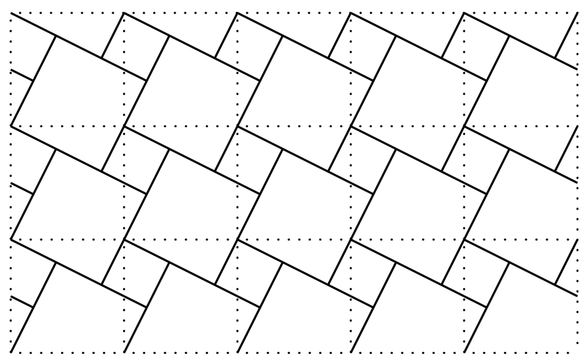
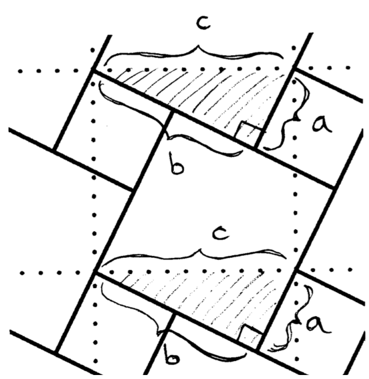
Problem 1. Here is a picture proof of the Pythagorean theorem:

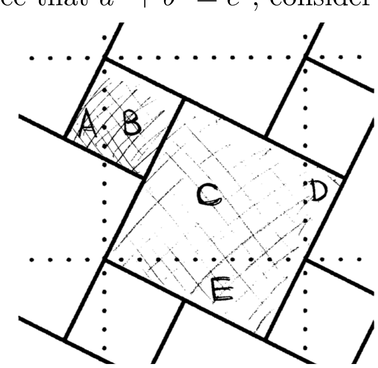


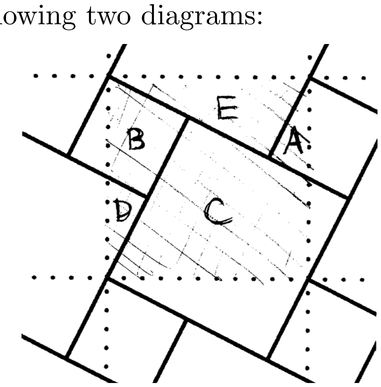
Your job is to **explain the proof** at an informal level. For example, imagine that you are tutoring a high school student. It might be helpful to label the triangle and the three side lengths, but please **don't use algebra**. [Hint: What is the area of the dotted square?]

Explanation. Let $a < b < c$ be the side lengths of the three squares in the tiling. The squares of sides a and b are solid, while the square of side c is dotted. The tiling contains many copies of a **right triangle** with these side lengths. Here are two copies of the triangle:



In order to see that $a^2 + b^2 = c^2$, consider the following two diagrams:





In the left hand diagram we have cut the two squares with side lengths a and b into **five pieces** with areas A, B, C, D, E, so that

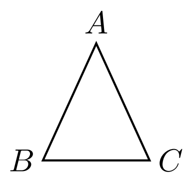
$$a^2 + b^2 = (A + B) + (C + D + E).$$

But in the right hand diagram we notice that the square of side length c can be cut into the **same five pieces**, so that

$$c^2 = A + B + C + D + E = a^2 + b^2.$$

□

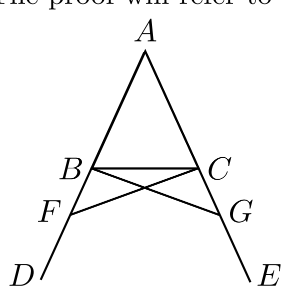
Problem 2. Proposition I.5 in Euclid has aquired the name *pons asinorum* (the "bridge of asses" or "bridge of fools"). Apparently, many students never got past this point in their studies. The proposition says the following: Consider a triangle $\triangle ABC$. If the side lengths $\overline{AB}$ and $\overline{AC}$ are equal, then the angles $\measuredangle ABC$ and $\measuredangle ACB$ are equal:



$$\boxed{\overline{AB} = \overline{AC} \quad \Longrightarrow \quad \measuredangle ABC = \measuredangle ACB}$$

(a) Look up Euclid's proof of Prop I.5 and try to understand it.
(b) **Write down the proof in your own words.** Your goal is to make the proof as understandable as possible. Maybe you can improve on Euclid.

Euclid's Proof of Proposition I.5. The proof will refer to the following diagram:



Start with the triangle $\triangle ABC$ and assume that side lengths $\overline{AB} = \overline{AC}$ are equal. First extend the edge AB to some random point D and extend the edge AC to some random point E [Postulate 2]. Choose a random point on the segment BD and call it F. [Why are we allowed to do this? I don't know.] Now find the point G on the segment AE such that the lengths $\overline{AF} = \overline{AG}$ are equal [Proposition I.3.] (Here we assumed that the segment AE is long enough, but we can always make it longer if necessary.) Draw segments CF and GB [Postulate 1]. Now observe that

$$\overline{AF} = \overline{AG}, \qquad \measuredangle FAC = \measuredangle GAB, \qquad \overline{AC} = \overline{AB}.$$

It follows from Proposition I.4 [the side-angle-side criterion] that the triangles $\triangle ACF$ and $\triangle ABG$ are **congruent** (i.e., have all angles and side lengths the same). In particular, this implies that

But now from the congruence of $\triangle ACF$ and $\triangle ABG$ we have:

- $\overline{BF} = \overline{AF} - \overline{AB} = \overline{AG} - \overline{AC} = \overline{CG}$, [Common Notions 1,3]

- $\measuredangle BFC = \measuredangle AFC = \measuredangle AGB = \measuredangle CGB$, [Common Notion 1]
- $\overline{CF} = \overline{BG}$.

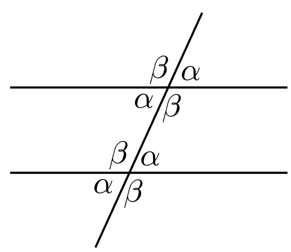
It follows again from Proposition I.4 [side-angle-side] that triangles $\triangle BCF$ and $\triangle CBG$ are congruent. In particular, this implies that $\measuredangle BCF = \measuredangle CBG$. Finally, from Common Notions 1 and 3 we conclude that

$$\measuredangle ABC = \measuredangle ABG - \measuredangle CBG = \measuredangle ACF - \measuredangle BCF = \measuredangle ACB,$$

as desired. □

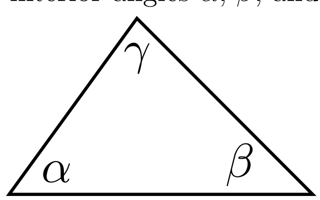
[Remark: Do you see why people find this proof difficult?]

Problem 3. Prove that the interior angles of any (Euclidean) triangle sum to 180°. You may use the following two facts without proof. **Prop I.31:** Given a line ℓ and a point p not on ℓ, **it is possible** to draw a line through p parallel to ℓ. **Prop I.29:** If a line falls on two parallel lines, then the corresponding angles are equal, as in the following figure:



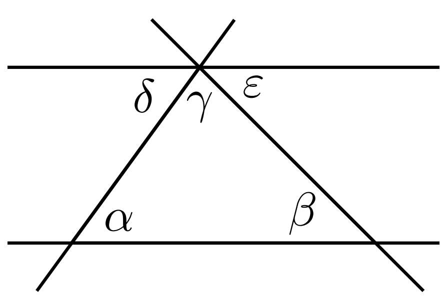
[Hint: Your proof should begin as follows: "Consider a triangle with interior angles α, β, γ. We will prove that $\alpha + \beta + \gamma = 180°$." Now draw the triangle.]

Proof. Consider a triangle with interior angles α, β, and γ:



We will prove that $\alpha + \beta + \gamma = 180°$.

To do this, we first use Proposition I.31 to draw a line through the vertex at angle γ that is **parallel** to the opposite side of the triangle. Then after extending the three sides of the triangle we obtain the following diagram:



The labeled angles δ and ε are initially unknown to us. However, by applying Proposition I.29 twice we find that $\delta = \alpha$ and $\varepsilon = \beta$. Since δ, γ, and ε form a straight line we must have $\delta + \gamma + \varepsilon = 180°$. Finally, we conclude that

$$\alpha + \beta + \gamma = \delta + \varepsilon + \gamma = 180°,$$

as desired. □

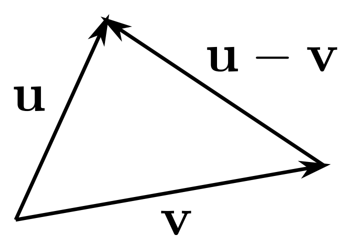
Problem 4. The *dot product* of the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is defined by $\mathbf{u} \bullet \mathbf{v} := u_1v_1 + u_2v_2$. The *length* $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ satisfies $\|\mathbf{u}\|^2 = \mathbf{u} \bullet \mathbf{u}$.

(a) The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ form the three sides of a triangle. Draw this triangle.
(b) Use algebra (not geometry) to prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v})$.
(c) Use the formula from part (b) to prove the following statement:

"the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \bullet \mathbf{v} = 0$."

[Hint: Remember your picture from part (a). You are allowed to assume that the Pythagorean Theorem and its converse are true.]

The triangle for part (a) looks like this:



(Since we discussed this in class, I imagine everyone's picture will look roughly the same.) Part (b) is simply a computation:

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}\|^2 &= \|(u_1, u_2) - (v_1, v_2)\|^2 \\
&= \|(u_1 - v_1, u_2 - v_2)\|^2 \\
&= (u_1 - v_1)^2 + (u_2 - v_2)^2 \\
&= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) \\
&= (u_1^2 + u_2^2) + (v_1^2 + v_2)^2 - 2(u_1v_1 + u_2v_2) \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).
\end{aligned}$$

Part (c) asks for a proof, so I'll write it nicely.

Proof. Consider any two vectors $\mathbf{u} = (u_1, v_1)$ and $\mathbf{v} = (v_1, v_2)$. We will prove that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (i.e., $\mathbf{u} \perp \mathbf{v}$) if and only if $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 = 0$. Since this is an "if and only if" statement we must prove each direction separately.

First we will prove that $\mathbf{u} \perp \mathbf{v}$ implies $\mathbf{u} \bullet \mathbf{v} = 0$. So let us assume that $\mathbf{u} \perp \mathbf{v}$. In this case, the triangle from part (a) is a right triangle and the Pythagorean Theorem tells us that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

On the other hand, we know from part (b) that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

Equating these two expressions for $\|\mathbf{u}-\mathbf{v}\|^2$ gives

$$\begin{aligned} \cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} &= \cancel{\|\mathbf{u}\|^2} + \cancel{\|\mathbf{v}\|^2} - 2(\mathbf{u} \bullet \mathbf{v}) \\ 0 &= -2(\mathbf{u} \bullet \mathbf{v}) \\ 0 &= \mathbf{u} \bullet \mathbf{v}, \end{aligned}$$

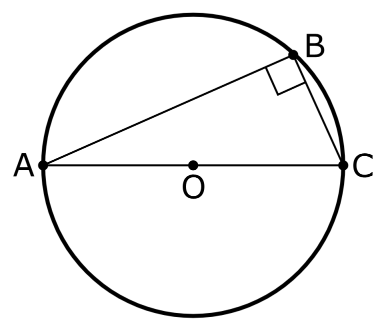
as desired.

Now we will prove that $\mathbf{u} \bullet \mathbf{v} = 0$ implies $\mathbf{u} \perp \mathbf{v}$. So let us assume that $\mathbf{u} \bullet \mathbf{v} = 0$. Then from part (b) we have

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(0) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2. \end{aligned}$$

Finally, by applying this fact to the triangle in part (a) we conclude from the **converse** of the Pythagorean Theorem that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $90°$. In other words, we conclude that $\mathbf{u} \perp \mathbf{v}$. □

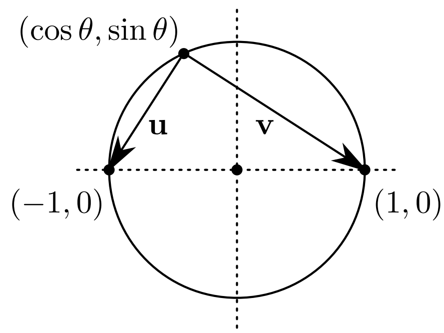
Problem 5. Let AC be the diameter of a circle and let B be any other point of the circle. Then I claim that $\measuredangle ABC$ is a right angle:



Legend says that this is the oldest theorem in the world, and that it was proved by Thales of Alexandria in the 6th century BC. You will give a modern ("analytic") proof.

You can assume that $O = (0,0)$ is at the origin of the Cartesian plane. You can also assume that $A = (-1,0)$ and $C = (1,0)$, so the circle has radius 1. Then the point B has the form $B = (\cos\theta, \sin\theta)$ for some angle θ. Now use Problem 4 to **prove that the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular**. [Hint: Head minus tail. You may use any trig identities that you know.]

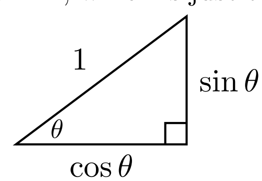
Proof. Suppose that our circle has radius 1 and is centered at the origin $(0,0)$ in the Cartesian plane. Suppose that $A = (-1,0)$ and $C = (1,0)$ and recall that any point B on the circle has the form $B = (\cos\theta, \sin\theta)$ for some angle θ. Furthermore, let $\mathbf{u} = \overrightarrow{BA}$ and $\mathbf{v} = \overrightarrow{BC}$. Then we have the following diagram:



Our goal is to prove that the vectors **u** and **v** are pendicular, and from Problem 4 it is enough to prove that the dot product is zero: $\mathbf{u} \bullet \mathbf{v} = 0$. To do this we first use the "head minus tail" rule to observe that

$$\begin{aligned}\mathbf{u} &= (-1,0) - (\cos\theta, \sin\theta) = (-1-\cos\theta, -\sin\theta),\\ \mathbf{v} &= (1,0) - (\cos\theta, \sin\theta) = (1-\cos\theta, -\sin\theta).\end{aligned}$$

And recall the identity $\cos^2\theta + \sin^2\theta = 1$, which is just the Pythagorean Theorem in disguise:



Finally, from the definition of the dot product we have

$$\begin{aligned}\mathbf{u} \bullet \mathbf{v} &= (1-\cos\theta, -\sin\theta) \bullet (-\cos\theta - 1, -\sin\theta)\\ &= (1-\cos\theta)(-1-\cos\theta) + (-\sin\theta)(-\sin\theta)\\ &= (-1+\cos^2\theta) + \sin^2\theta\\ &= -1 + (\cos^2\theta + \sin^2\theta)\\ &= -1+1\\ &= 0,\end{aligned}$$

as desired. $\square$