Introduction

So, what is MTH 230 about?

"Abstract Math"

Fine, but what does that mean?

It means that we will focus on the ideas behind mathematics instead of on computations and problem solving.

We will spend a lot of time learning to express mathematics rigorously.

"Rigor = Clarity + Precision"

We will learn to write mathematics in full sentences in such a way that we will be understood.

There is an art to this and it takes practice; that's why we have a whole class devoted to it.

Abstract math uses a lot of jargon.
We will see words such as:

- axiom
- lemma
- theorem
- corollary
- proof

Q: What is a "theorem"?

A: A theorem is a mathematical statement that has been demonstrated to be true.

I'm sure you've seen some theorems before. Name some . . .

For example, let's consider the oldest and most important theorem in mathematics:

<u>The Pythagorean Theorem.</u>

Draw squares on the sides of a triangle. Let A, B, C be their areas. Let θ be the angle opposite the square with area C.

The Pythagorean Theorem says the following:

$$\boxed{\text{If } \theta = 90^\circ \text{ then } A + B = C.}$$

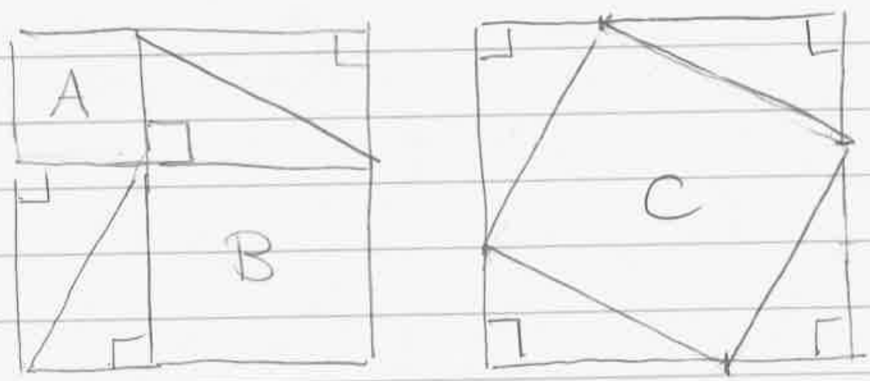
Why is this true?
Is it true?

To qualify as a theorem it must have a convincing demonstration (i.e., a "proof"). I'll try to convince you.

Proof: Let's assume that $\theta = 90°$. In this case we want to show that

$$A + B = C.$$

To see this, we consider the following two squares:



These two squares have equal area and each contains 4 copies of the original triangle. If we remove the 4 triangles then what remains on each side must still have equal area. Hence

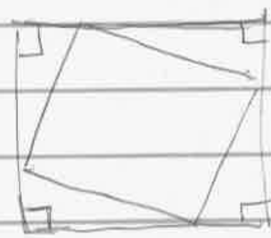
$$A + B = C.$$

$\blacksquare$

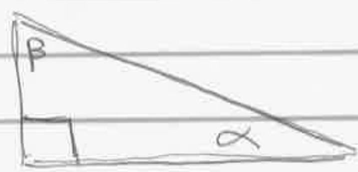
Are you convinced?
(You don't have to say yes.)

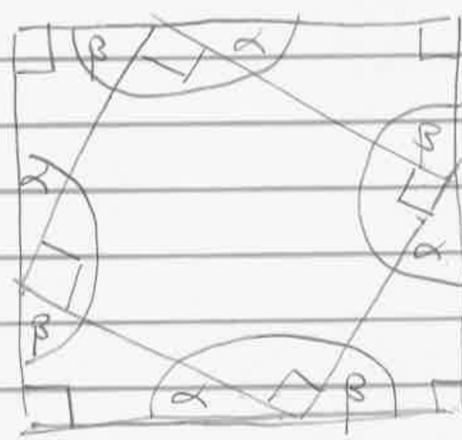
There are many possible questions (complaints) you could have.

Possible complaint:

Why is  a square?

We need to check that the edges are straight lines.





Is it true that $\alpha + \beta + 90^\circ = 180^\circ$?

Yes, because the angles in a triangle always sum to 180°.

Now are you convinced?
(You don't have to say yes.)

You might ask <u>why</u> the angles in a triangle sum to 180°.

If I can't convince you then the proof is not valid. If you are very skeptical (stubborn) then I will have a hard time finishing the proof.

Here is a schematic diagram:

(If $\theta = 90°$ then $A + B = C$)

$\Uparrow$

(angles in a triangle sum to 180°)

$\Uparrow$

maybe you have more complaints...

When can I stop?!

Answer: At some point I will "just stop".

Pythagorean Theorem

$\Uparrow$

Angles in a triangle sum to 180°

$\Uparrow$

$\vdots$

$\Uparrow$

$\boxed{\text{AXIOMS}}$

The proof ends when we reach some "axioms". These are true statements that are supposed to be "self-evident" (i.e., need no proof).

If you still don't agree, that's

Your Problem !

//

The "axiomatic method" just discussed was invented in a very specific time and place:

Miletus, Asia Minor, ~600 BC.

The first person to use the method was apparently Thales of Miletus (c. 625 BC – c. 546 BC). This way of thinking was central to Ancient Greek thought and reached its full expression with Euclid of Alexandria (~300 BC) in his work called "The Elements"

The Axiomatic Method
and
Euclid's Elements

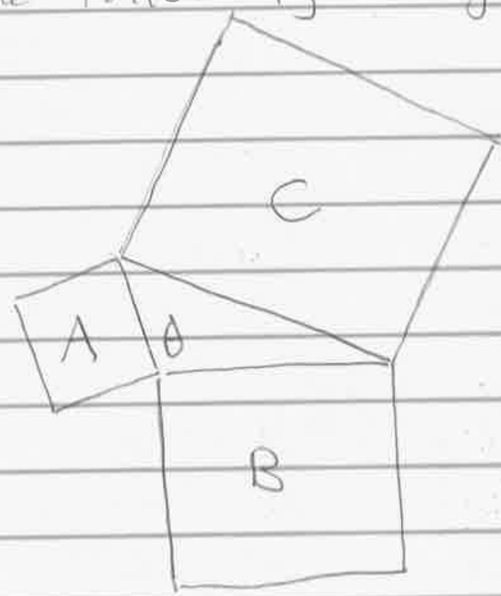
Recall from last time:

- A "theorem" is a mathematical statement that has been demonstrated to be true.
- A convincing demonstration is called a "proof".

Whether a proof is actually convincing might depend on the audience. Last time I tried to convince you that the Pythagorean Theorem is true.

Here is a careful statement of the theorem:

Consider a triangle and draw squares on its sides. Let the squares have areas A, B, C and let θ be the angle opposite the square of area C, as in the following diagram:



In this case I claim that

$$\boxed{\text{if } \theta = 90^\circ \text{ then } A + B = C.}$$

[Was all of that preamble necessary? Yes, if you want to be polite.]

Then I gave a proof and you tried to poke holes in it.

We agreed that the proof was convincing
if the angles in a triangle sum to 180°:

Pythagorean Theorem

$\Uparrow$

Angles in a triangle sum to 180°.

$\Uparrow$

?

But it is certainly fair to ask why the angles sum to 180°. To complete the proof I must convince you of this. But now I'm worried that the proof will never end. [What if you keep asking questions forever?]

The Ancient Greeks came up with a solution to this problem called the

"axiomatic method".

Idea: We will agree beforehand on a set of "axioms". These are supposed to be self-evidently true statements (i.e., they don't need to be proved). Then to prove a theorem we will show that it follows logically from the axioms.

Pythagorean Theorem

$\Uparrow$

Angles in a triangle sum to $180°$

$\Uparrow$

$\vdots$

$\Uparrow$

$\boxed{\text{AXIOMS}}$

As soon as we do this the proof is done. [If you still don't agree, that's "your problem".]

The first set of axioms were written down by Euclid of Alexandria (~300 BC) in a work called "The Elements".

Euclid had 10 axioms, divided into 5 "postulates" and 5 "common notions".

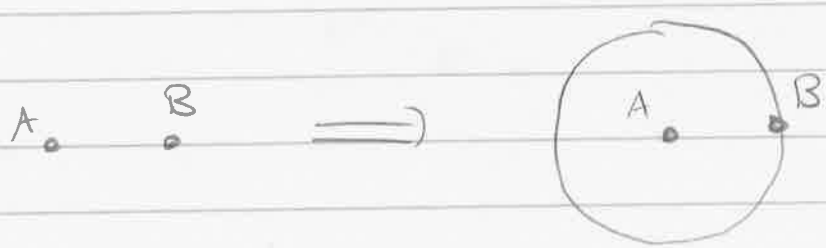
[See handout.]

The postulates are the basic facts about geometric constructions.

P1

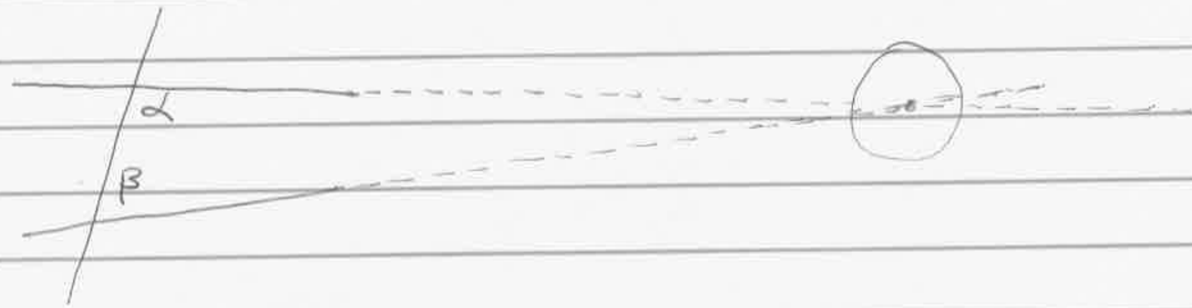
P2

P3



(P4) All right angles are equal to each other.
[What does he mean by this ?]

(P5)



If $\alpha + \beta < 180^\circ$ then the two lines will eventually meet on this side.

The common notions (CN1) – (CN5) describe properties of comparison. In modern terms we would use the symbols

"=" and "<"

And that's all. With a lot of work Euclid was able to rigorously prove all the theorems of Ancient Greek mathematics from these 10 axioms.

AMAZING!

Today we distinguish different kinds of mathematical truths with different names.

"Lemma": A technical result that is not intrinsically interesting, but we will need it later.

"Theorem": A substantial result that is intrinsically interesting.

"Corollary": An interesting observation that follows easily from a theorem.

[There are more, but I won't bore you with them.].

Euclid didn't distinguish; he just called them all "propositions".

The Elements contains XIII Books and 468 propositions.

↓

Book I has 48 Propositions. It is basically a proof of the Pythagorean Theorem, which is the subject of the final two theorems.

⋮

Book $\overline{XIII}$ is a construction of the five regular ("platonic") solids. The hardest one to construct is the dodecahedron.

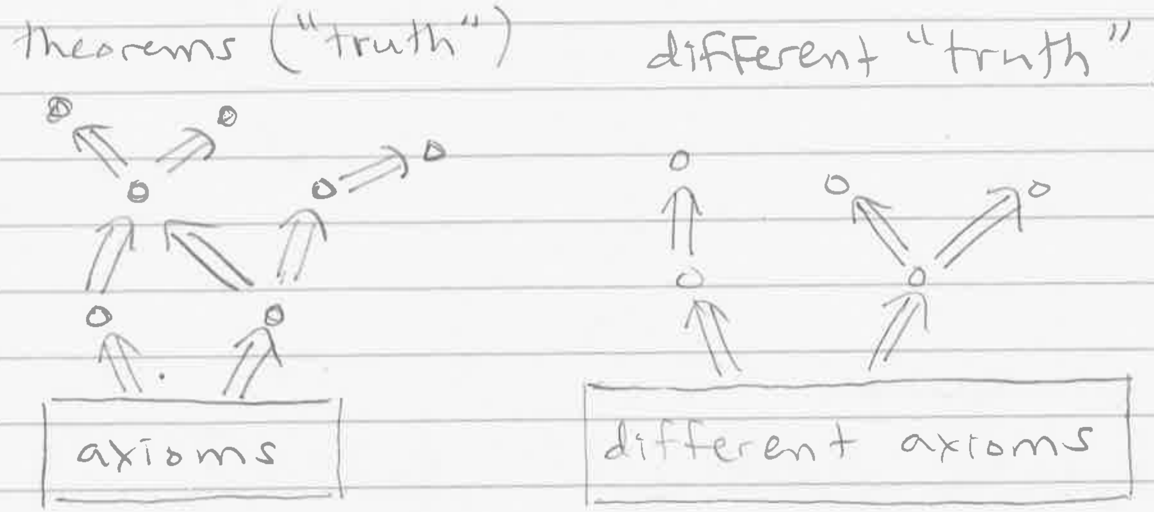
[Remark: The regular solids were associated with the basic kinds of atoms in Plato's "Timaeus".

tetrahedron	=	fire
cube	=	earth
octahedron	=	air
icosahedron	=	water
dodecahedron	=	aether.

Perhaps that's why Euclid called his work "The Elements".]

Examples From Euclid

The structure of mathematics is like a tree. Here's a schematic diagram:



- Theorems are "true" statements that can be deduced logically from the axioms.
- Different axioms; different notion of "truth"

A proof looks like this:

- A sequence of logical deductions leading back to the axioms.

Last time I described Euclid's "Elements" (~300 BC) which was the first axiomatic system.

Today we'll see what a Euclidean style proof looks like.

Book I begins with

- 23 Definitions
- 5 Postulates
- 5 Common Notions

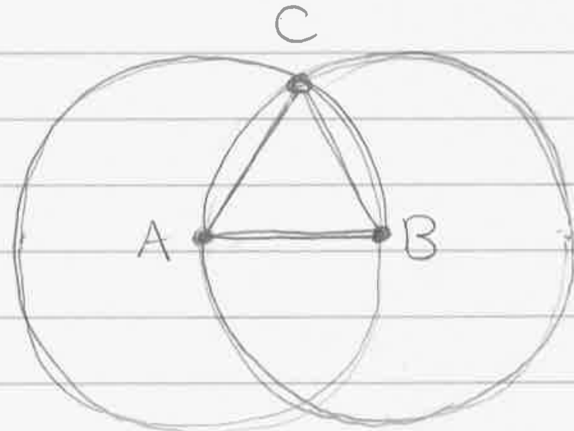
"axioms"

Then Euclid proceeds to prove a sequence of 48 Propositions ("theorems").

Here is the very first one.

Prop I.1: Given two points A and B, we can construct a point C such that the triangle $\triangle ABC$ is <u>equilateral</u> (all three sides have the same length).

Proof:



Let A, B be the given points and connect AB with a line. P1

Draw circle with center A and radius AB. Draw circle with center B and radius AB. P3 P3

Let C be a point of intersection of the two circles. ?

Draw the line segments AC & BC (P1).

Since C is on both of the circles we have $\overline{AC} = \overline{AB}$ and $\overline{BC} = \overline{AB}$ (Def I.15)

Hence we also have $\overline{AC} = \overline{BC}$ (CN1)

We conclude that ΔABC is equilateral as desired. (Def I.20)

Q.E.D.

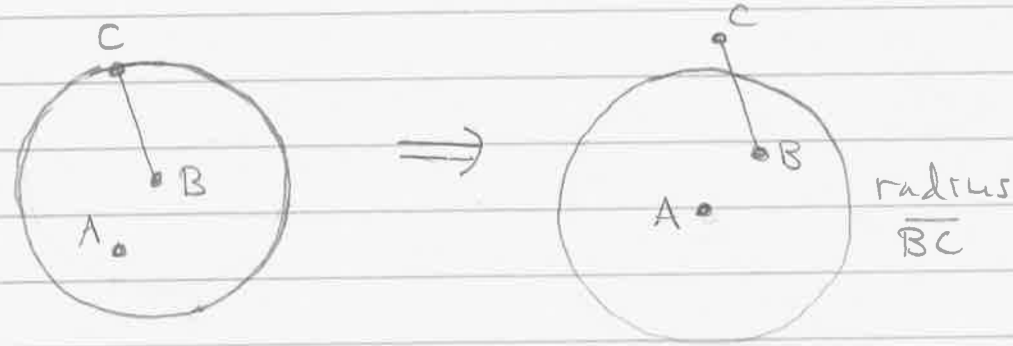
Oops! Why does the point C exist? Euclid forgot to give a justification for this (maybe he thought it was too obvious).

Remark: David Hilbert "Fixed" The Elements in 1899. He needed 20 axioms instead of 10!

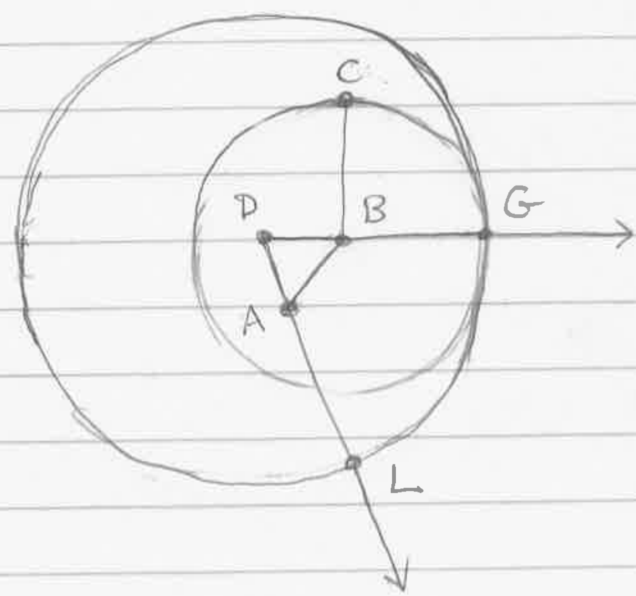
Here is the second proposition.

Prop I.2 : To move a circle.

Given a circle with center B and radius BC and another point A, we can construct a circle with center A and radius of length $\overline{BC}$.



Proof :



Draw equilateral triangle $\triangle ABD$ Prop I.1

Extend DB to G. P2

Construct circle with center D
and radius DG. P3

Extend DA to L. P2

We have $\overline{DL} = \overline{DG}$ Def I.15
and $\overline{AD} = \overline{BD}$ Def I.20

Hence $\overline{DL} - \overline{AD} = \overline{DG} - \overline{BD}$ CN 3
$\overline{AL} = \overline{BG}$

But we know that $\overline{BG} = \overline{BC}$ Def I.15

Since $\overline{AL} = \overline{BG}$ and $\overline{BG} = \overline{BC}$
we conclude that $\overline{AL} = \overline{BC}$ CN 1
as desired.

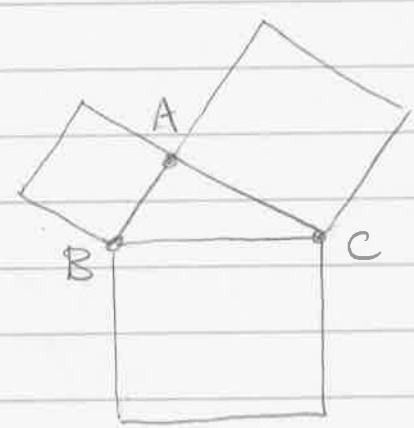
Q.E.D.

Q: Why didn't Euclid just include it as an axiom that you can pick up the compass and move it?

A: Because he didn't need to! The idea is to keep the number of axioms as small as possible.

It continues like this for a long while.

Prop I.47 is the Pythagorean Theorem:



If $\measuredangle BAC = 90^\circ$ then we have

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 .$$

But Book I has 48 propositions. What could Prop I.48 possibly be?!

Prop I.48 is the "converse" of the Pythagorean Theorem:

If $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$ then $\angle BAC = 90°$.

Isn't that just the same thing? NO.

[Given logical statements P and Q we will write "$P \Rightarrow Q$" to mean "P implies Q", or "if P then Q".

It is important to note that the "converse" statements

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P$$

are NOT logically equivalent. Can you think of some examples?]

The converse of the Pyth. Thm. does not <u>need</u> to be true, it just <u>happens</u> to be true.

Euclid's Proof of the Pythagorean Theorem

Last time we discussed the first two propositions of Euclid's Book I.

Recall: Book I has 48 propositions.

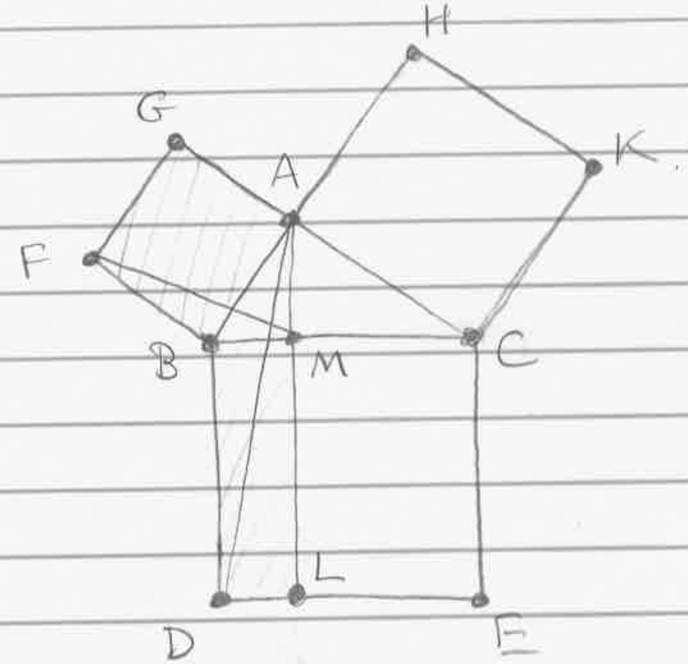
Today we'll discuss Propositions 47 & 48.
[We'll skip Propositions 3 — 46.].

Proposition I.47 :

Let ΔABC be a right-angled triangle, where $\measuredangle BAC$ is the right angle. In this case we claim that

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 .$$

Proof: This proof is accompanied by Euclid's famous "windmill diagram":



Let $\angle BAC = 90°$

Construct squares on the three sides with vertices labeled as in the diagram. [This is allowed by Proposition I.46]

Since $\angle BAG = 90°$ [Def I.22] and $\angle BAC = 90°$ [by assumption] we conclude that CA and AG form a straight line. [Prop I.14]

For the same reason, BA and AH form a straight line.

Since $\measuredangle DBC = \measuredangle FBA$ [by Def I.22 and CN1 (and P4 ?)]. Adding $\measuredangle ABC$ to both gives

$$\measuredangle DBC + \measuredangle ABC = \measuredangle FBA + \measuredangle ABC \quad \text{CN2}$$
$$\measuredangle DBA = \measuredangle FBC.$$

Since $\overline{DB} = \overline{BC}$ and $\overline{FB} = \overline{BA}$ [Def I.22] we conclude that the triangles $\triangle ABD$ and $\triangle FBC$ are congruent [by Prop I.4 (side-angle-side criterion)].

Now draw a line through A parallel to either BD or CE [by Prop. I.31] and extend it to M and L.

The parallelogram BDLM has twice the area of triangle $\triangle ABD$, and the square FBAG has twice the area of triangle $\triangle FBC$ [by Prop I.41] because they have the same bases and are between the same parallels.

Since ΔABP and ΔFBC are congruent we have

$$\text{area}(\Delta ABD) = \text{area}(\Delta FBC)$$
$$2 \cdot \text{area}(\Delta ABD) = 2 \cdot \text{area}(\Delta FBC).$$
$$\text{area}(BDLM) = \text{area}(FBAG).$$
$$\text{area}(BDLM) = \overline{AB}^2 .$$

[Some Common Notions were used here.].

Using a similar argument we can show that

$$\text{area}(CELM) = \overline{AC}^2 .$$

Finally, we have

$$\begin{aligned} \overline{BC}^2 &= \text{area}(BDEC) \\ &= \text{area}(BDLM) + \text{area}(CELM) \\ &= \overline{AB}^2 + \overline{AC}^2 . \end{aligned}$$

[Again some Common Notions were used.]

Q.E.D.

Remark: Some people think that this proof of the Pythagorean Theorem is due to Euclid himself and that is why he gives it such a prominent place in The Elements. It is basically the climax of Book I. But there is one more proposition after it.

Question: What could Prop I.48 possibly be? What comes after the Pythagorean Theorem?

Proposition I.48:

Consider a triangle ΔABC and suppose that the side lengths satisfy

$$\overline{BC}^2 = \overline{AB}^2 + \overline{BC}^2 .$$

In this case we claim that $\measuredangle BAC$ is a right angle.

Principle of the Contrapositive

Last time we carefully went through Euclid's proof of Prop I.47 (the Pythagorean Theorem). It was quite involved.

[See the handout showing the tree leading from I.47 back to the axioms.]

But Book I has 48 propositions. If Prop I.47 was the climax, then what is Prop I.48?

Prop I.48: Let ΔABC be a triangle. <u>If</u> the side lengths satisfy $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$, <u>then</u> $\measuredangle BAC$ is a right angle.

Wait, isn't that just the same as Prop I.47?

NO!

Let me define some logical notation.

Given logical statements P and Q we will write "$P \Rightarrow Q$" to mean that "P implies Q" or "if P then Q".

It is important to note that the two statements

"$P \Rightarrow Q$" and "$Q \Rightarrow P$"

are NOT logically equivalent. [We call these statements the "converses" of each other.]

Can you think of some examples?

If $P \Rightarrow Q$ is a true statement, then $Q \Rightarrow P$ <u>may or may not</u> be true. If it is true then it will require a separate proof.

We have a notation for this. We write "$P \Leftrightarrow Q$" to mean that "$P \Rightarrow Q$ and $Q \Rightarrow P$". In words we can say that "P implies Q and Q implies P", but there is a shorter way.

Note that

"$Q \Rightarrow P$" = "P if Q"

"$P \Rightarrow Q$" = "P only if Q"

Thus we can say

"$P \Leftrightarrow Q$" = "P <u>if and only</u> if Q"

= "P iff Q"

Example: Let ΔABC be a triangle and consider the statements

$$P = \text{“} \measuredangle BAC = 90^\circ \text{”}$$
$$Q = \text{“} \overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 \text{”}.$$

Prop I.47 claims that $P \Rightarrow Q$ and Prop I.48 claims that $Q \Rightarrow P$.

We can put the two propositions together by saying

$$P \Leftrightarrow Q.$$

But note that this requires two <u>separate proofs</u>. I encourage you to look up Euclid's proof of Prop I.48. It's much shorter than his proof of Prop I.47.

Example: Let n be an integer. I claim that

$$n \text{ is even} \Leftrightarrow n^2 \text{ is even.}$$

How can we prove this?

Let's think about it before we launch into a proof. We will need to prove two separate statements

(1) n is even $\Longrightarrow$ n^2 is even
(2) n is odd $\Longrightarrow$ n^2 is odd.

The first one is is not so bad.

Definition: We say that n is even if there exists an integer k such that

$$n = 2k.$$

If n is even then we have $n = 2k$ for some integer k. Then we have

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $n^2 = 2$(some integer) we conclude that n^2 is even.

We have proved (1) ✓

How can we prove (2) ?

Suppose that n^2 is even. In this case we want to prove that n is even.

Since n^2 is even, there exists an integer k such that $n^2 = 2k$.

Then we have $n = ?$

Hmm... we're STUCK.

We will need a trick.

Given a logical statement P, we write $\neg P$ (and we say "not P") for the opposite statement.

☆ The Principle of the Contrapositive:

Given logical statements P and Q, the statements

"$P \Rightarrow Q$" and "$\neg Q \Rightarrow \neg P$"

are logically equivalent. ///

[We will justify this later. For now let's just use it.]

Instead of proving

$$n^2 \text{ even} \implies n \text{ even}$$

we will prove the contrapositive statement

$$n \text{ odd} \implies n^2 \text{ odd}.$$

So suppose that n is odd, i.e., we have $n = 2k+1$ for some integer k [we'll justify this later.]. Then we have

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2(\text{some integer}) + 1. \end{aligned}$$

Hence n^2 is odd.

We have proved (2) ✓

Pythagoras Today = The Dot Product

We have seen that mathematical proof was based on Euclid's axioms for thousands of years. But we don't use them anymore.

Today mathematics is mostly based on axioms for number systems. We'll see how that works later, but for now:

"How can we use number axioms to prove theorems about geometry?"

This is based on the idea of Cartesian coordinates. After making this translation we will find that

The Euclidean Plane

becomes

A 2-dimensional real vector space with a "dot product".

Here's how we do geometry today:

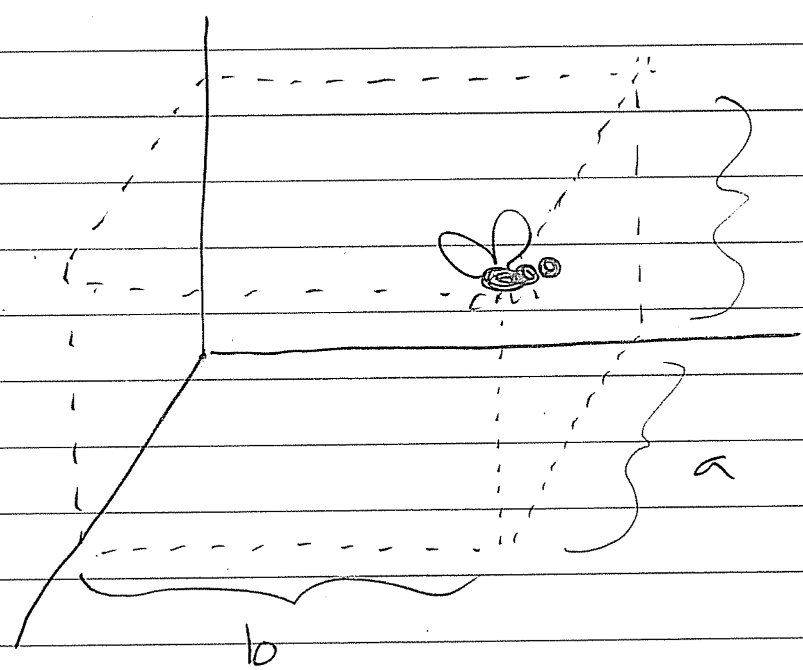
Q: What is "space"?
What is a "point"?

Answer (Fermat, Descartes ~1637):

A point is an ordered list of numbers

What?!

Descartes was lying in bed.
He saw a fly in the corner.



Imagine a rectangular box with dimensions a, b, c (in some order).

Descartes realized that the numbers (a, b, c) uniquely specify the <u>position</u> of the fly!

(a, b, c) = the "(Des)Cartesian coordinates" of the fly.

Actually, we'll write it like this:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{v}$$

We'll call it a <u>vector</u>.
A vector is just an ordered list of numbers.

Q: Why is this useful?

point = list of numbers.

A: Because we can do <u>algebra</u> with numbers!

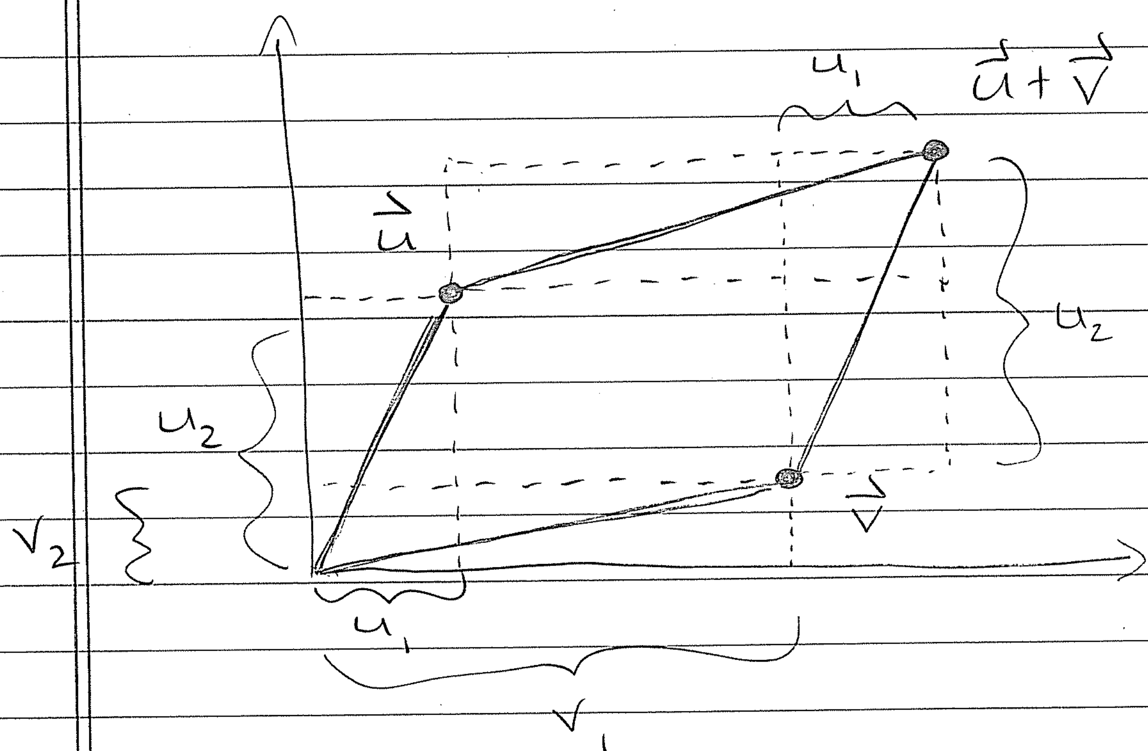
Example: We can add vectors

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$\vec{u} + \vec{v} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}.$$

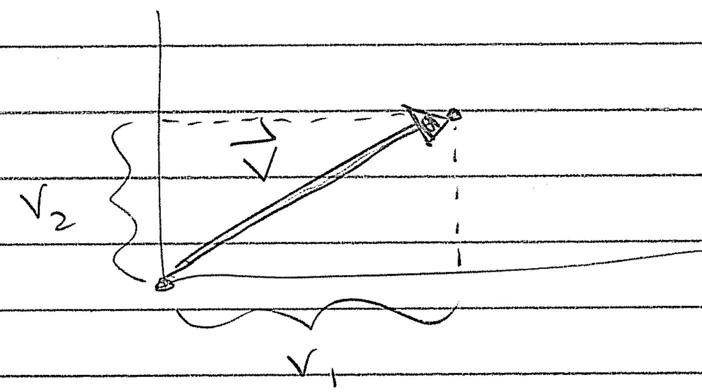
definition   add "componentwise".

What does it mean?

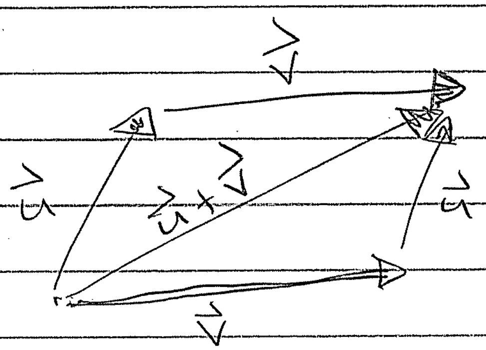


"Parallelogram Law of Vector Addition"

Sometimes useful to think of $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ as an <u>arrow</u> with head at $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and tail at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



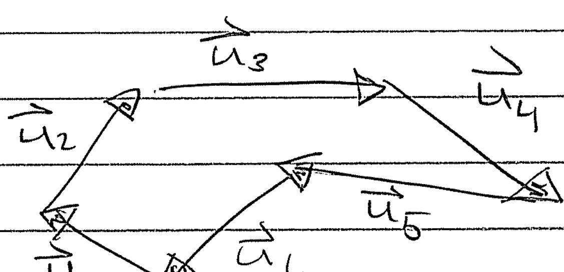
Then adding vectors is easy:



Order doesn't matter

$\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

Vectors add head-to-tail.



$$\vec{u}_1+\vec{u}_2+\vec{u}_3+\vec{u}_4+\vec{u}_5+\vec{u}_6 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$$

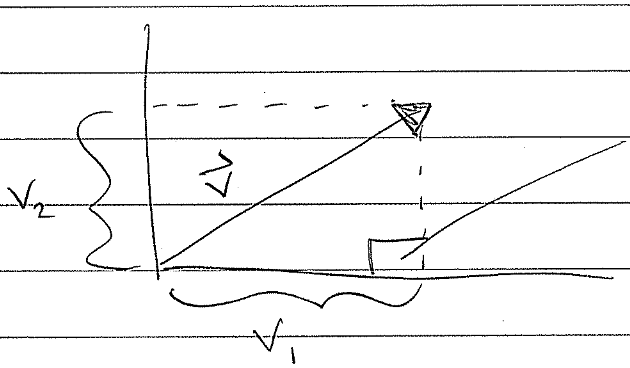
"zero vector"

Two Perspectives :

vector is a point $\longleftrightarrow$ vector is an arrow.

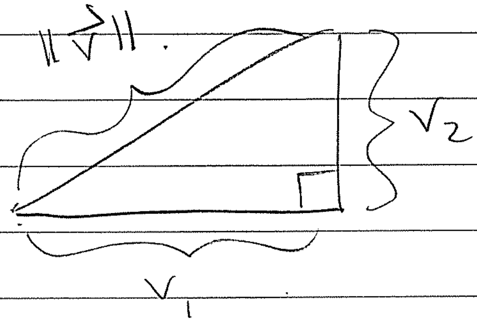
Switch back and forth when we want, using the point of reference $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (the "origin").

Q: What is the <u>length</u> of a vector?



right angle (90°).

See the triangle?



Let $\|\vec{v}\| :=$ length of arrow $\vec{v}$.

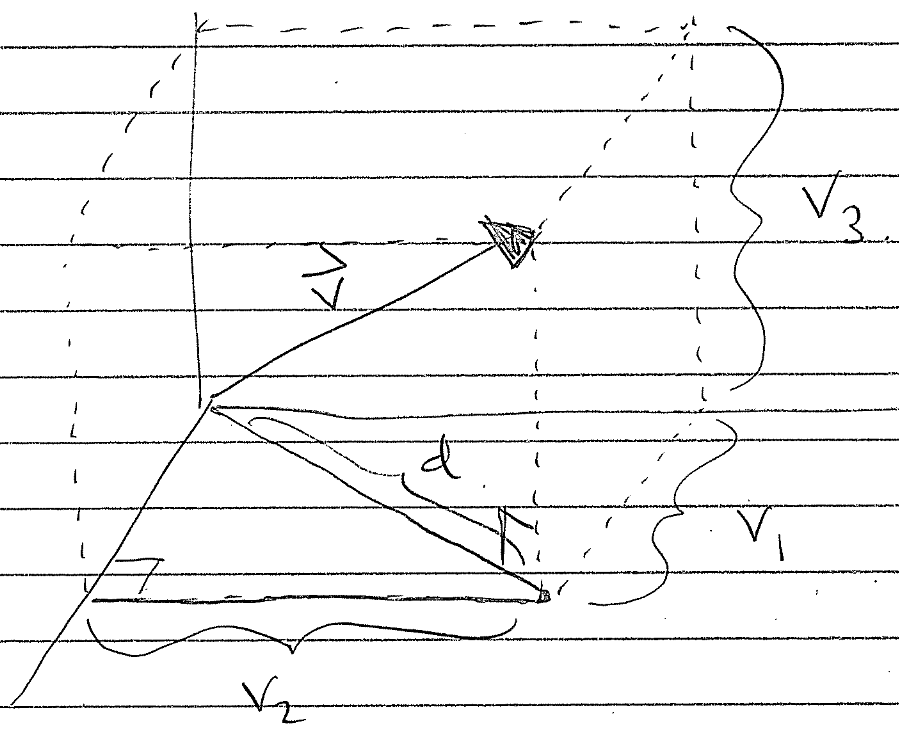
Pythagoras says:

$$\|\vec{v}\|^2 = v_1^2 + v_2^2$$

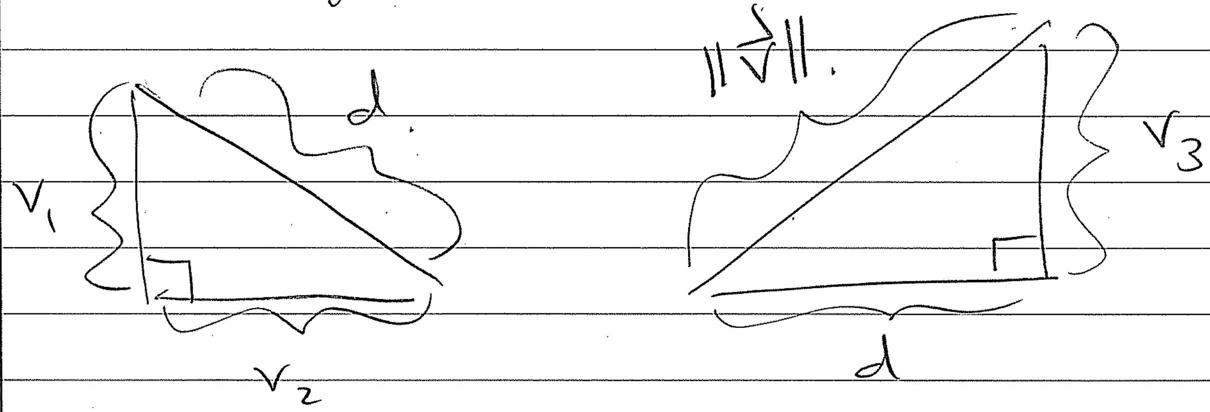
$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

What about in 3D? Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\|\vec{v}\| = ?$$



Two triangles



Pythagoras says:

$$d^2 = v_1^2 + v_2^2 \qquad \|\vec{v}\|^2 = d^2 + v_3^2$$

$$\begin{aligned}\|\vec{v}\|^2 &= d^2 + v_3^2 \\ &= v_1^2 + v_2^2 + v_3^2\end{aligned}$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Question: If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$,

is it true that

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \quad ??$$

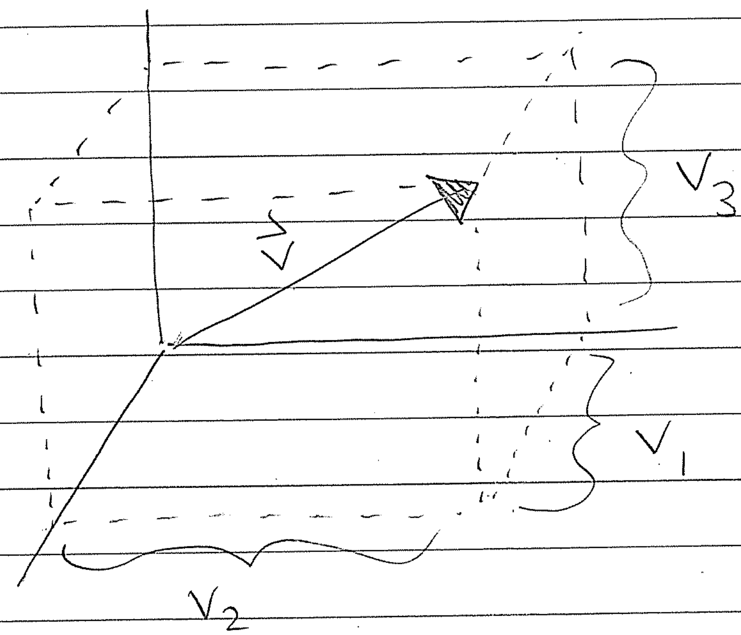
Answer: Sure, why not?

Recall :

A vector is an ordered list of numbers

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

OR an arrow in space (Descartes).



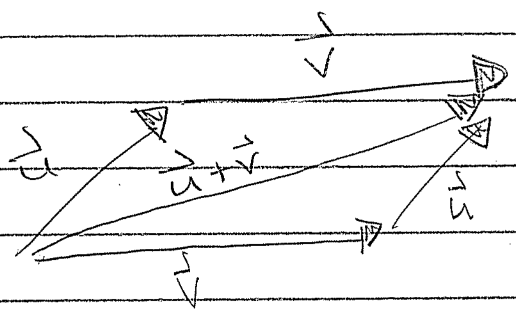
Vectors can be added.

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ then

$$\vec{u} + \vec{v} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

definition

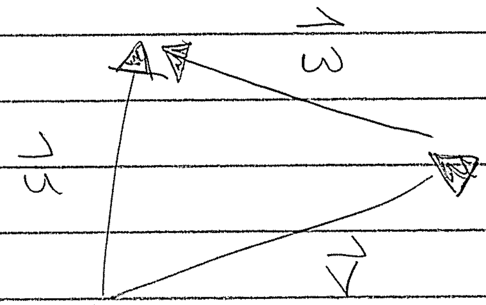
Geometrically, arrows are added head-to-tail:



$\vec{u}+\vec{v} = \vec{v}+\vec{u}$

Q: Can we subtract vectors?

Consider:

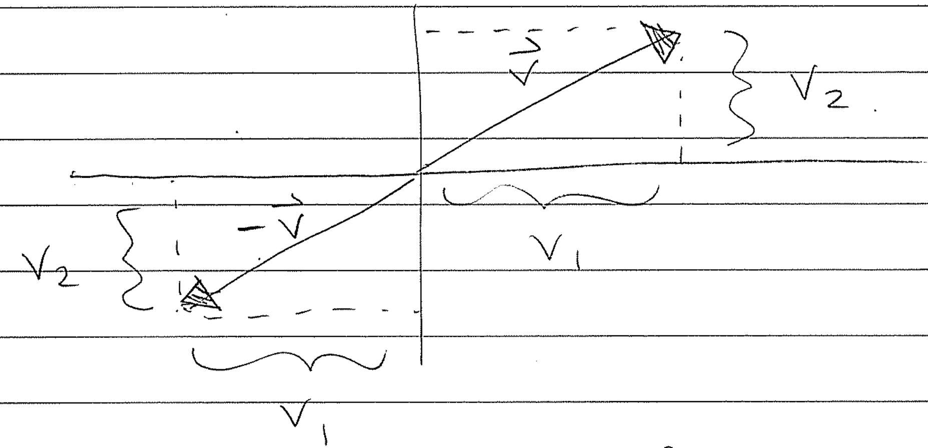


$\vec{v}+\vec{w} = \vec{u}$

We would like to say $\vec{w} =$ "$\vec{u}-\vec{v}$"

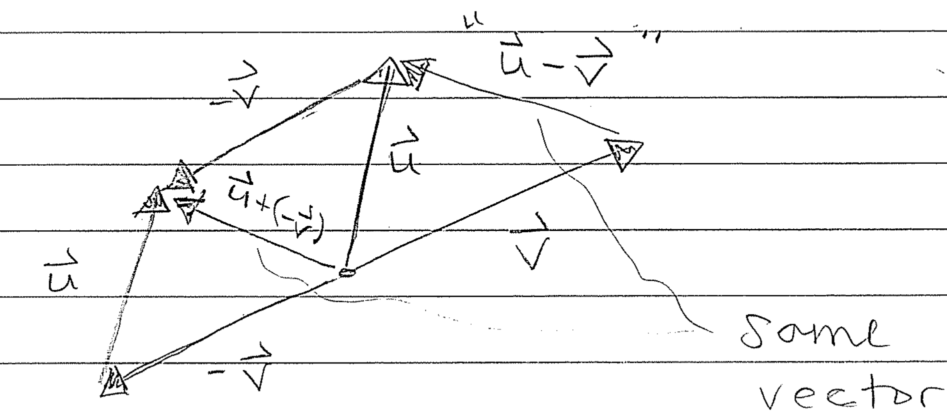
Does this make sense? Yes!

Given arrow $\vec{v}$, let $-\vec{v}$ be the same arrow with opposite direction.



We write $-\vec{v} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$

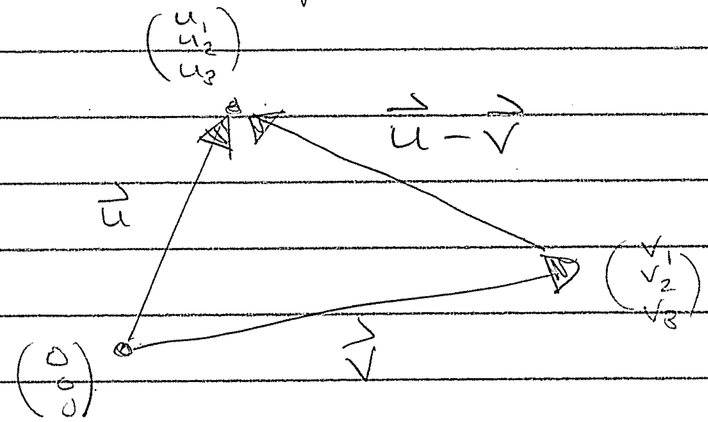
Then observe:



By definition we have

$$\vec{u} - \vec{v} := \vec{u} + (-\vec{v})$$

This allows us to compute the distance between two points.



The distance $\text{dist}(\vec{u}, \vec{v})$ is the length of the arrow

$$\vec{u} - \vec{v} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{pmatrix}$$

Recall Pythagoras:

$$\text{dist}(\vec{u}, \vec{v})^2 = \|\vec{u} - \vec{v}\|^2$$

$$= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2.$$

Keep Going ...

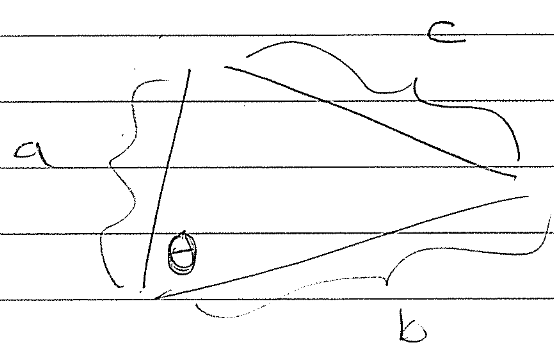
$$= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)$$

$$= (u_1^2 + u_2^2 + u_3)^2 + (v_1^2 + v_2^2 + v_3^2)$$

$$\quad - 2(u_1v_1 + u_2v_2 + u_3v_3).$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\underbrace{(u_1v_1 + u_2v_2 + u_3v_3)}_{\text{what is this ?}}.$$

Think about a general triangle.



Ancient "Law of Cosines" says

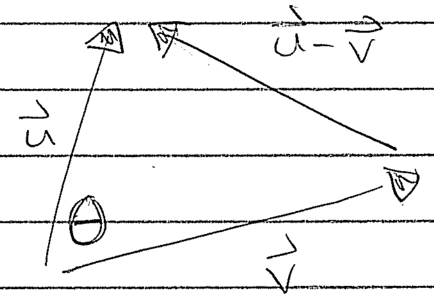
$$c^2 = a^2 + b^2 - \text{something}$$

$$= a^2 + b^2 - 2ab\cos\theta.$$

When $\cos\theta = 0$ (i.e. $\theta = 90°$) this is the Pythagorean Theorem.

$$c^2 = a^2 + b^2.$$

Now recall our triangle of vectors



We computed

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\left(u_1 v_1 + u_2 v_2 + u_3 v_3\right).$$

Law of cosines says

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

In conclusion, we have

$$u_1v_1 + u_2v_2 + u_3v_3 = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta.$$

This is very useful. If we define the dot product of vectors

$$\boxed{\vec{u} \circ \vec{v} := u_1v_1 + u_2v_2 + u_3v_3}$$

then we can measure the distance between any two points:

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \circ (\vec{u} - \vec{v})$$

and we can measure the angle between any two vectors:

$$\text{angle}(\vec{u}, \vec{v}) = \cos^{-1}\left(\frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

$$= \cos^{-1}\left(\frac{\vec{u} \circ \vec{v}}{\sqrt{\vec{u} \circ \vec{u}} \cdot \sqrt{\vec{v} \circ \vec{v}}}\right).$$

What more do we need?